TAYLOR POLYNOMIALS

1. Find the fourth Taylor polynomial of e^x at $x = 0$

Solution: We have $f(x) = e^x$. Then

$$\begin{aligned} f'(x) = e^x &\implies f'(0) = 1 \\ f''(x) = e^x &\implies f''(0) = 1 \\ f^{(3)}(x) = e^x &\implies f^{(3)}(0) = 1 \\ f^{(4)}(x) = e^x &\implies f^{(4)}(0) = 1 \end{aligned}$$

Then

$$e^x \approx 1 + 1(x-0) + \frac{1}{2}(x-0)^2 + \frac{1}{6}(x-0)^3 + \frac{1}{24}(x-0)^4 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

close to $x = 0$

2. Find the fourth Taylor polynomial of $\ln(x)$ at $x = 1$

Solution: We have $f(x) = \ln(x)$. Then

$$\begin{aligned} f'(x) = \frac{1}{x} &\implies f'(1) = 1 \\ f''(x) = -\frac{1}{x^2} &\implies f''(1) = -1 \\ f^{(3)}(x) = \frac{2}{x^3} &\implies f^{(3)}(1) = 2 \\ f^{(4)}(x) = -\frac{6}{x^4} &\implies f^{(4)}(1) = -6 \end{aligned}$$

Then

$$\ln(x) \approx 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{24}(x-1)^4$$

close to $x = 1$

3. Find the third Taylor polynomial of $\frac{1}{1-x}$ at $x = 0$

Solution: We have $f(x) = \frac{1}{1-x}$. Then

$$\begin{aligned} f'(x) &= \frac{0 \cdot (1-x) - 1 \cdot (-1)}{(1-x)^2} = \frac{1}{(1-x)^2} \implies f'(0) = 1 \\ f''(x) &= \frac{0 \cdot (1-x)^2 - 1 \cdot 2 \cdot (1-x) \cdot (-1)}{(1-x)^4} = \frac{2}{(1-x)^3} \implies f''(0) = 2 \\ f^{(3)}(x) &= \frac{0 \cdot (1-x)^3 - 2 \cdot 3 \cdot (1-x)^2 \cdot (-1)}{(1-x)^6} = \frac{6}{(1-x)^4} \implies f^{(3)}(0) = 6 \end{aligned}$$

Then

$$\frac{1}{1-x} \approx 1 + 1(x-0) + \frac{2}{2}(x-0)^2 + \frac{6}{6}(x-0)^3 = 1 + x + x^2 + x^3$$

close to $x = 0$

4. Find the seventh Taylor polynomial approximation of $\sin(x)$ at $x = 0$

Solution: We have $f(x) = \sin(x) \implies f(0) = 0$. Then

$$f'(x) = \cos(x) \implies f'(0) = 1$$
$$f''(x) = -\sin(x) \implies f''(0) = 0$$
$$f^{(3)}(x) = -\cos(x) \implies f^{(3)}(0) = -1$$
$$f^{(4)}(x) = \sin(x) \implies f^{(4)}(0) = 0$$
$$f^{(5)}(x) = \cos(x) \implies f^{(5)}(0) = 1$$
$$f^{(6)}(x) = -\sin(x) \implies f^{(6)}(0) = 0$$
$$f^{(7)}(x) = -\cos(x) \implies f^{(7)}(0) = -1$$

Then

$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

close to $x = 0$.

5. Find the fifth Taylor polynomial approximation of $\sqrt{x}$ at $x = 1$

Solution: We have $f(x) = \sqrt{x} \implies f(1) = 1$. For simplicity, we write $f(x) = x^{1/2}$, as it will help us to find the derivatives faster. Then

$$f'(x) = \frac{1}{2}x^{-1/2} \implies f'(1) = \frac{1}{2}$$
$$f''(x) = \frac{-1}{4}x^{-3/2} \implies f''(1) = -\frac{1}{4}$$
$$f^{(3)}(x) = \frac{3}{8}x^{-5/2} \implies f^{(3)}(1) = \frac{3}{8}$$
$$f^{(4)}(x) = \frac{-15}{16}x^{-7/2} \implies f^{(4)}(1) = -\frac{15}{16}$$
$$f^{(5)}(x) = \frac{15 \cdot 7}{32}x^{-9/2} \implies f^{(5)}(1) = \frac{15 \cdot 7}{32}$$

Then

$$\sqrt{x} \approx 1 + \frac{1}{2}(x-1) - \frac{1}{4 \cdot 2}(x-1)^2 + \frac{3}{8 \cdot 3!}(x-1)^3 - \frac{15}{16 \cdot 4!}(x-1)^4 + \frac{15 \cdot 7}{32 \cdot 5!}(x-1)^5$$

close to $x = 1$.