PRACTICE PROBLEMS

(1) Find the vertical and horizontal asymptotes of the following functions:

(a) $f(x) = \dfrac{x^2 - x - 6}{x^2 - x - 20}$

Solution: The horizontal asymptote is given by $\lim_{x\to\infty} \dfrac{x^2 - x - 6}{x^2 - x - 20} = 1$ (since we have the same power of x in both numerator and denominator, the limit is given by the ratio of the coefficents in front of the highest power of x, which is x^2), so $y = 1$ is a horizontal asymptote.
Using the quadratic formula, we factor the numerator and denominator

$$f(x) = \frac{x^2 - x - 6}{x^2 - x - 20} = \frac{(x-3)(x+2)}{(x-5)(x+4)}$$

and the vertical asymptotes are given by $x = 5, x = -4$.

(b) $g(x) = \dfrac{x+1}{(x+3)(x+5)}$

Solution: The horizontal asymptote is given by $\lim_{x\to\infty} \dfrac{x+1}{(x+3)(x+5)} = 0$ (since we have larger powers on the denominator), so $y = 0$ is a horizontal asymptote.
The vertical asymptotes are given by $x = -3, x = -5$.

(c) $h(x) = \dfrac{(x+1)^2}{x^2 + 4x + 3}$

Solution: The horizontal asymptote is given by

$$\lim_{x\to\infty} \frac{(x+1)^2}{x^2+4x+3} = \lim_{x\to\infty} \frac{x^2+2x+1}{x^2+4x+3} = 1,$$

so $y = 1$ is a horizontal asymptote.
Using the quadratic formula, we factor the denominator

$$h(x) = \frac{(x+1)^2}{(x+1)(x+3)} = \frac{x+1}{x+3} \qquad \text{when } x \neq 1$$

and the vertical asymptote is given by $x = -3$ ($x = -1$ is not a vertical asymptote; instead, the function has a removable discontinuity at $x = -1$)

(2) On what intervals are the following functions continuous?

(a) $\arctan\left(-x^2 + \dfrac{5}{x} - \sqrt{x+1}\right)$

Solution: As a composition of inverse trig, root and rational functions, $\arctan\left(-x^2 + \frac{5}{x} - \sqrt{x+1}\right)$ is continuous on its domain.
Since arctangent is defined on all real numbers, the domain for our functions is given by $x \neq 0$ and $x + 1 \geq 0$, Thus, $x \geq -1, x \neq 0$, and the domain is given by $[-1, 0) \cup (0, +\infty)$.

(b) $\ln\left(\dfrac{\sqrt{x+2}}{x}\right)$

Solution: As a combination of logarithmic, root and rational functions, the function is continuous on its domain.
The input in the natural log has to be positive, so $\frac{\sqrt{x+2}}{x} > 0$. Since the square root is always positive, we need the denominator to be positive, so we have $x > 0$. In addition, to take square roots we need $x + 2 \geq 0$, so $x \geq -2$. But we already need $x > 0$, so the domain is $(0, +\infty)$.

(c) $5x\sqrt{x^2 + x}$

Solution: Again, the function is contnuous on its domain.
For this we need $x^2 + x \geq 0$. But $x^2 + x = x(x + 1)$, and for the product to be positive, either both factors need to be positive, or both negative.
Case 1: $x \geq 0$ and $x + 1 \geq 0$, so $x \geq 0$.
Case 2: $x \leq 0$ and $x + 1 \leq 0$, so $x \leq -1$.
Therefore, the function is defined on $(-\infty, -1] \cup [0, +\infty)$, and it is continuous there as well.

(d) $\dfrac{\sqrt{x+1} - \sqrt{x-1}}{3x}$

Solution: Again, the function is contnuous on its domain.
For this we need $x + 1 \geq 0,\ x - 1 \geq 0$ and $x \neq 0$. This is the same as

$$x \geq -1,$$

$$x \geq 1,$$

and also

$$x \neq 0.$$

For all three conditions to happen, $x \geq 1$, so the domain is $[1, +\infty)$ and the function is continuous on this interval.

(3) In general, 4th degree polynomials don't have to have a root (e.g. $f(x) = x^4 + 1$). Show that $g(x) = 4x^4 - 10x^3 + 4x^2 - 6x - 10$ has a root.

Solution: We have $f(0) = -10, f(1) = 4-10+4-6-10 = -18$, so we cannot use the Intervediate Value Theorem yet. However, $f(-1) = 4+10+4+6-10 = 14$. Since $f(-1) > 0, f(0) < 0$, by the Intermediae Value Theorem there must be a number c between -1 and 0 such that $g(c) = 0$, so there must be a root between -1 and 0.

(4) Find the following limits:

(a) $\lim\limits_{x\to 5} \dfrac{x^2 - 2x - 15}{x - 5}$

Solution: We cannot plug in $x = 5$, since the expression is not defined at $x = 5$ (division by 0). So we factor the numerator:

$$\lim_{x\to 5} \frac{x^2 - 2x - 15}{x - 5} = \lim_{x\to 5} \frac{(x-5)(x+3)}{x-5} = \lim_{x\to 5}(x+3) = 8.$$

(b) $\lim_{x\to 1} \frac{x^2-3}{x+5}$

Solution: Since we have a rational function defined at $x = 1$, we plug in $x = 1$:

$$\lim_{x\to 1} \frac{x^2-3}{x+5} = \frac{1^2-3}{1+5} = \frac{-1}{3}.$$

(c) $\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x}$

Solution: We cannot plug in $x = 0$, since the expression is not defined at $x = 0$ (division by 0), and furthermore plugging in $x = 0$ gives a limit of the form $\frac{0}{0}$, which we cannot compute. Instead we rationalize the numerator:

$$\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x} = \lim_{x\to 0} \frac{(\sqrt{9+x}-3)(\sqrt{9+x}+3)}{x(\sqrt{9+x}+3)} = \lim_{x\to 0} \frac{9+x-9}{x(\sqrt{9+x}+3)} =$$
$$\lim_{x\to 0} \frac{1}{\sqrt{9+x}+3} = \frac{1}{\sqrt{9+0}+3} = \frac{1}{6}.$$

(d) $\lim_{x\to\pi} \sin(x+\sin(x))$

Solution: Since $\sin(x)$ is continuous on all of the real line, we can just plug in $x = \pi/2$:

$$\lim_{x\to\pi} \sin(x+\sin(x)) = \sin(\pi+\sin(\pi)) = \sin(\pi+0) = \sin(\pi) = 0$$

(5) Do the following sequences converge? If so, to what?

(a) $a_n = \frac{n}{n^3+1}$

Solution: The sequence converges to 0, since we have higher powers of n in the denominator than in the numerator.

(b) $b_n = \frac{n^3+5}{n^2+3n+4}$

Solution: The sequence does not converge, since we have smaller powers of n in the denominator than in the numerator.

(c) $c_n = \frac{(-3)^n}{6^n}$

Solution: The sequence converges to 0, since $\frac{(-3)^n}{6^n} = \left(\frac{-3}{6}\right)^n = \left(\frac{-1}{2}\right)^n$, and the sequence $\{r^n\}$ converges to 0 for r in the interval $(-1, 1)$.

(d) $d_n = \cos(n\pi/2)$

Solution: The sequence does not converge: its terms are $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$, and the values do not stabilize to one particular limit in the long run.