Practice problems review IV

Exercise 1: sequences a) For each of the following sequences, determine if it converges or not. If it does, what is its limit?

b) For each of the following sequences, determine if it is bounded or not. If it is bounded, find a bound.

(1) $a_n = \arcsin\left(\frac{1}{n}\right)$, $n \geq 1$

Solution: Converges to 0: since $\lim_{n\to\infty} \frac{1}{n} = 0$, $\lim_{n\to\infty} \arcsin\left(\frac{1}{n}\right) = \arcsin(0) = 0$. Since $-1 \leq \arcsin(x) \leq 1$, the sequence is bounded. One bound would be 2, since then $-2 < \arcsin(x) < 2$.

(2) $b_n = \left(\frac{101}{100}\right)^n$, $n \geq 1$

Solution: Does not converge and is not bounded: $\frac{101}{100} > 1$, so the powers grow very large, e.g. $(\frac{101}{100})^{1000} > 20959$

(3) $c_n = e^{\frac{n}{n+1}}$, $n \geq 1$

Solution: Converges to e: since $\lim_{n\to\infty} \frac{n}{n+1} = 1$ (same powers of n in numerator and denominator), $\lim_{n\to\infty} e^{\frac{n}{n+1}} = e^1 = e$. Since the sequence converges, it has to be bounded; to find such a bound observe $-1 < \frac{n}{n+1} < 1$ (the sequence $\frac{n}{n+1}$ is increasing, and its terms are $\{1/2, 2/3, 3/4, 4/5, \dots\}$, getting closer and closer to 1 but never reaching 1). Then $e^{-1} < e^{\frac{n}{n+1}} < e^1$, so a bound would be e: $-e < e^{\frac{n}{n+1}} < e$.

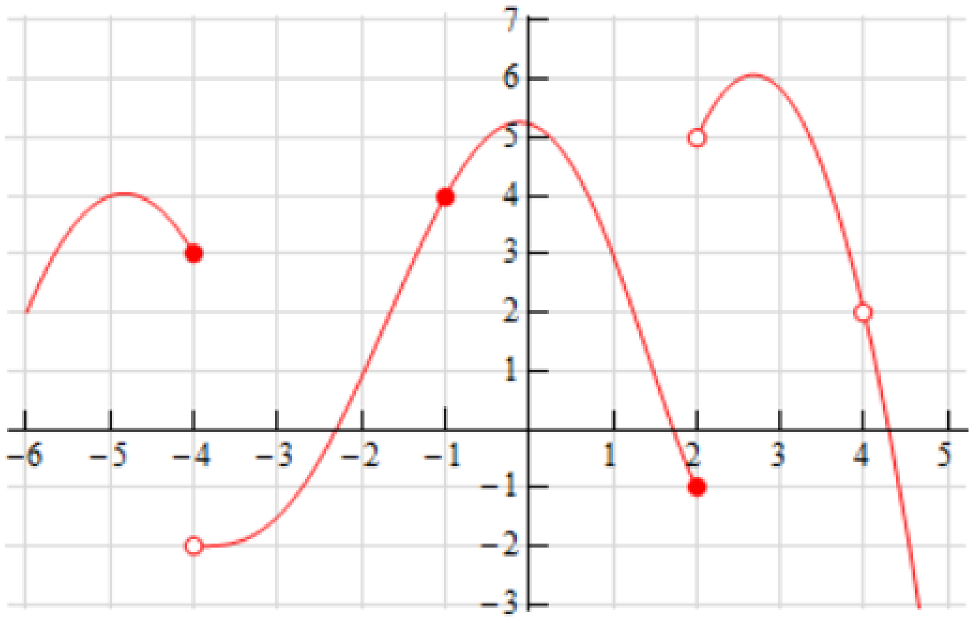
Exercise 2: continuity and limits For each of the following functions, determine

I. where it is discontinuous, and what types of discontinuities those are

II. the intervals on which the function is continuous

III. $\lim_{x\to\infty} f\left(\frac{x^2+x}{x^2+5}\right)$

(1) If $f(x)$ is given by the following graph



Solution: I. The function is discontinuous at $x = -4, x = 2$, and $x = 4$. At $x = -4$ and $x = 2$ we have a jump discontinuity; at $x = 4$ we have a removable discontinuity.

II. The function is continuous on the intervals $(-\infty, -4) \cup (-4, 2) \cup (2, 4) \cup (4, \infty)$.

III. Since $\lim_{x\to\infty} \frac{x^2+x}{x^2+5} = 1$, then $\lim_{x\to\infty} f\left(\frac{x^2+x}{x^2+5}\right) = f(1) = 3$

(2) If f is defined by the following equations:

$$f(x) = \begin{cases} \sin(x) & \text{if } x < 0 \\ 5 & \text{if } x = 0 \\ \dfrac{x}{(x-2)(x-3)} & \text{if } x > 0 \end{cases}$$

Solution: I. As a piecewise defined function, we need to check for discontinuities at the end of the piecewise intervals, as well as at the points that are not in its domain. Since $\lim_{x\to 0^-} f(x) = \sin(0) = 0$ but $f(0) = 5$, we have a discontinuity at $x = 0$. Since $\frac{x}{(x-2)(x-3)}$ is not defined at $x = 2, x = 3$, we have two discontinuities there as well. At $x = 0$ we have a removable discontinuity, since both one-sided limits are the same at 0: $\lim_{x\to 0^-} f(x) = \sin(0) = 0$ and $\lim_{x\to 0^+} f(x) = \dfrac{0}{(0-2)(0-3)} = 0$.

At $x = 2, x = 3$ we have an infinite discontinuity, since there we have vertical asymptotes.

II. The function is continuous on the intervals $(-\infty, 0) \cup (0, 2) \cup (2, 3) \cup (3, +\infty)$

III. Since $\lim_{x\to\infty} \frac{x^2+x}{x^2+5} = 1$, and f is continuous at 1, then $\lim_{x\to\infty} f\left(\frac{x^2+x}{x^2+5}\right) = f(1) = 3 = \dfrac{1}{(1-2)(1-3)} = \dfrac{1}{2}$

Exercise 3: trig and inverse trig. Compute the following values as expressions without trig and inverse trig functions:

(1) $\arctan(-\sqrt{3}) = -\pi/3$, since $\tan(pi/3) = \sqrt{3}$

(2) $\arcsin(\sin(-\frac{3\pi}{4}) = \arcsin(-\frac{\sqrt{2}}{2}) = -\frac{\pi}{4}$ We can't use the cancellation laws since $-\frac{3\pi}{4}$ is not in the range of $\arcsin(x)$

(3) $\tan(\arccos(8/10)) = 6/8$ (draw triangle)

(4) $\sin(\arctan(\frac{x+4}{x+3})) = \frac{x+4}{\sqrt{2x^2+14x+25}}$, since the hypothenuse is given by Pythagorean theorem $\sqrt{(x+3)^2 + (x+4)^2} = \sqrt{2x^2 + 14x + 25}$ (draw triangle if not convinced)