Practice problems review III

Exercise 1: sequences For each of the following sequences, determine if it converges or not. If it does, find the limit.

(1) $a_n = \dfrac{1+x^2}{(2x+1)(3x+2)}$

Solution: $a_n = \dfrac{1+n^2}{(2n+1)(3n+2)} = \dfrac{n^2+1}{6n^2+7n+2}$

Since both numerator and denominator have the same largest power of n (which is n^2), the sequence converges to the quotient of the coefficients in front of n^2, and $\lim_{n\to\infty} a_n = \dfrac{1}{6}$.

(2) $b_n = \sin\left(\dfrac{\pi}{2} + \dfrac{1}{n}\right)$

Solution: The sequence converges.

Since $\lim_{n\to\infty}\left(\dfrac{\pi}{2} + \dfrac{1}{n}\right) = \dfrac{\pi}{2} + 0 = \dfrac{\pi}{2}$, $\lim_{n\to\infty} b_n = \sin\left(\dfrac{\pi}{2}\right) = 1$.

(3) $c_n = \dfrac{(\frac{-3}{2})^n}{(\frac{4}{5})^n}$

Solution: The sequence does not converge.

Since $c_n = \dfrac{(\frac{-3}{2})^n}{(\frac{4}{5})^n} = \dfrac{\frac{(-3)^n}{2^n}}{\frac{4^n}{5^n}} = \dfrac{(-3)^n}{2^n} \cdot \dfrac{5^n}{4^n} = \dfrac{-15^n}{8^n} = \left(\dfrac{-15}{8}\right)^n$, and $\dfrac{-15}{8} < -1$, the sequence does not converge (r^n would only converge for $-1 < r \leq 1$)

Exercise 2: trig and inverse trig. Evaluate each of the following:

(1) $\frac{d}{dx}(\cos(x))$ at $x = -\frac{\pi}{4}$

Solution: $\frac{d}{dx}(\cos(x)) = -\sin(x)$, and $-\sin(-\frac{\pi}{4}) = -(-\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}$

(2) $\arccos(\cos(-\frac{\pi}{2}))$

Solution: We cannot use the cancellation laws since $-\frac{\pi}{2}$ is not in the range of arccos, which is $[0, \pi]$. Instead, we evaluate step by step:

$$\arccos(\cos(-\frac{\pi}{2})) = \arccos(0) = \frac{\pi}{2}.$$

(3) $\frac{d}{dx}(x\arctan(x))$ at $x = -1$

Solution: $\frac{d}{dx}(x\arctan(x)) = (x)'\arctan(x) + x(\arctan(x))' = \arctan(x) + \frac{x}{1+x^2}$. Then

$$\arctan(-1) + \frac{-1}{1+(-1)^2} = -\frac{\pi}{4} - \frac{1}{2}$$

Exercise 3: Limits and continuity of functions. Let $f(x) = \ln(\frac{2x+4}{3x})$. Determine the following:

(1) On what intervals is $f(x)$ continuous?

Solution: As a combination of rational and logarithmic functions, f is continuous on its domain. Due to the log, the domain is given by

$$\frac{2x+4}{3x} > 0$$

Multiplying both sides by 3/2 we get

$$\begin{aligned} \frac{x+2}{x} &> 0 \\ \frac{x}{x} + \frac{2}{x} &> 0 \\ 1 + \frac{2}{x} &> 0 \\ \frac{2}{x} &> -1 \end{aligned}$$

If $x < 0$, then we need $2 < -x$, so $x < -2$. If $x > 0$, then we need $2 > -x$, so $x > -2$. In the latter case, we have both $x > 0$ and $x > -2$, so $x > 0$. Thus, the domain is $(-\infty, -2) \cup (0, +\infty)$.

(2) Find $\lim_{x\to\infty} f(x)$.

Solution: Since $\lim_{x\to\infty} \frac{2x+4}{3x} = \frac{2}{3}$, $\lim_{x\to\infty} \ln\left(\frac{2x+4}{3x}\right) = \ln\left(\frac{2}{3}\right)$.

(3) Find $f'(x)$. What is the domain of $f'(x)$?

Solution: Let $g(x) = \ln(x)$ and $h(x) = \frac{2x+4}{3x}$. Then $g'(x) = \frac{1}{x}$, $h'(x) = \frac{2(3x)-(2x+4)3}{(3x)^2} = \frac{-12}{9x^2} = -\frac{4}{3x^2}$. Then

$$f'(x) = g'(h(x))h'(x) = \frac{3x}{2x+4} \cdot \frac{-4}{3x} = \frac{-2}{(x+2)x}$$

Since the derivative gives the slope of the tangent line at a point $(x, f(x))$ on the graph of $f(x)$, the domain of the derivative is contained in the domain of $f(x)$. Thus, the domain of $f'(x)$ is contained in $(-\infty, -2) \cup (0, +\infty)$, and since $\frac{-2}{(x+2)x}$ is defined for all such values, the domain of $f'(x)$ is all of $(-\infty, -2) \cup (0, +\infty)$.

(4) Find $\lim_{x\to\infty} f'(x)$.

Solution: $\lim_{x\to\infty} \frac{-2}{(x+2)x} = 0$, since the denominator has larger powers of x.