Practice problems review II

Exercise 1: domains and ranges; inveses

(1) Let $f(x) = x^2 + 4, g(x) = \sqrt{x-4}$. What are the domains and ranges of $f, g, f \circ g$ and $g \circ f$?

Solution: The domain of f is $(-\infty, +\infty)$, the range is $[4, +\infty)$. The domain for g is $[4, +\infty$, the range is $[0, +\infty)$.

$(f \circ g)(x) = (\sqrt{x-4})^2 + 4$, and the domain is $[4, +\infty)$. Since $(f \circ g)(x) = (\sqrt{x-4})^2 + 4 = (x-4) + 4 = x$ given $x \geq 4$, the range is $[4, +\infty)$.

$(g \circ f)(x) = \sqrt{x^2+4-4} = \sqrt{x^2}$, and the domain is $(-\infty, +\infty)$. Since $(g \circ f)(x) = \sqrt{x^2} = |x|$, the range is $[0, +\infty)$.

(2) Let $f(x)$ have the domain $[2, 5]$ and range $[-3, 4]$. What are the domains and ranges of

(a) $f(-x+5) + 2$

Solution: Since f only accepts values between $[2, 5]$ as an input,

$$2 \leq -x + 5 \leq 5$$

$$-3 \leq -x \leq 0$$

$$0 \leq x \leq 3$$

so the domain is $[0, 3]$.
For the range, we know the output of f is between $[-3, 4]$, so

$$-3 \leq f(-x+5) \leq 4$$

$$-1 \leq f(-x+5) + 2 \leq 6$$

the range is $[-1, 6]$.

(b) $3f^{-1}(2x+1)$

Solution: We are dealing with f^{-1}, not f, so the domain of f^{-1} is $[-3, 4]$ and the range is $[2, 5]$. Since f^{-1} only accepts values between $[-3, 4]$ as an input,

$$-3 \leq 2x + 1 \leq 4$$

$$-4 \leq 2x \leq 3$$

$$-2 \leq x \leq 1.5$$

so the domain is $[-2, 1.5]$.
For the range, we know the output of f^{-1} is between $[2, 5]$, so

$$2 \leq f^{-1}(2x+1) \leq 5$$

$$6 \leq 3f^{-1}(2x+1) \leq 15$$

the range is $[6, 15]$.

(c) $2f(3x+1)+2$

Solution: Since f only accepts values between $[2,5]$ as input,

$$2 \leq 3x+1 \leq 5$$
$$1 \leq 3x \leq 4$$
$$1/3 \leq x \leq 4/3$$

so the domain is $[1/3, 4/3]$.
For the range, we know the output of f is between $[-3,4]$, so

$$-3 \leq f(3x+1) \leq 4$$
$$-6 \leq 2f(3x+1) \leq 8$$
$$-4 \leq 2f(3x+1)+2 \leq 10$$

the range is $[-4, 10]$.

(3) Let $f(x) = 1+\sqrt{2+3x}$. What are the domain and range of f? Find a formula for $f^{-1}(x)$. What are the domain and range of $f^{-1}(x)$?

Solution: The domain of f is $[-2/3, +\infty)$. The range is $[1, +\infty)$. To find the inverse, we first solve for x in terms of y:

$$y = 1+\sqrt{2+3x}$$
$$y-1 = \sqrt{2+3x}$$
$$2+3x = (y-1)^2$$
$$3x = (y-1)^2 - 2$$
$$x = \frac{(y-1)^2-2}{3}.$$

Now we relabel, and $f^{-1}(x) = \frac{(x-1)^2-2}{3}$. The domain of f^{-1} is the range of f, namely $[1,+\infty)$, and the range of f^{-1} is the domain of f, namely $[-2/3, +\infty)$.

Exercise 2: logarithmic and exponential equations Solve for x:

(1) $\ln(x+1)+\ln(x+2) = \ln(30)$

Solution: First, this equation is defined only when both $x+1>0$ and $x+2>0$, so $x>-1$. Then

$$\ln(x+1)+\ln(x+2) = \ln(30)$$
$$\ln[(x+1)(x+2)] = \ln(30)$$
$$e^{\ln[(x+1)(x+2)]} = e^{\ln(30)}$$
$$(x+1)(x+2) = 30$$
$$x^2+3x+2 = 30$$
$$x^2+3x-28 = 0$$

By the quadratic formula, we can factor

$$(x-4)(x+7) = 0$$

So the possible solutions are $x=4, x=-7$. Since at the beginning we were only looking for solutions $x>-1$, our only solution is $x=4$.

(2) $2^{5^x} = 5^{2^x}$

Solution: The equation is deined for all real x. We take $\log_2$ of both sides:

$$\log_2(2^{5^x}) = \log_2(5^{2^x})$$

$$5^x = 2^x \log_2 5$$

Now we take $\log_5$ of both sides:

$$\log_5(5^x) = \log_5(2^x \log_2 5)$$

$$x = \log_5(2^x) + \log_5(\log_2 5)$$

$$x = x \log_5 2 + \log_5(\log_2 5)$$

$$x - x \log_5 2 = \log_5(\log_2 5)$$

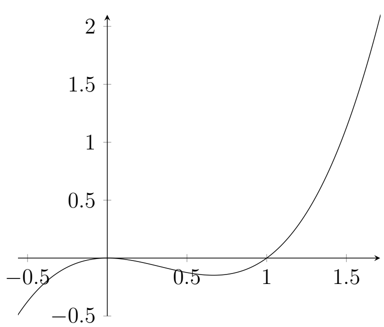
$$x(1 - \log_5 2) = \log_5(\log_2 5)$$

$$x = \frac{\log_5(\log_2 5)}{1 - \log_5 2}$$

This is our solution.

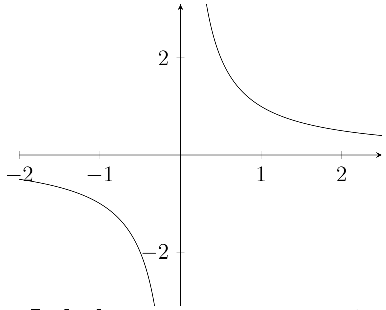
Exercise 3: library of functions Consider the following 4 classes of functions: linear, polynomial, rational, power. For each of the following graphs, write down which classes (if any) it belongs to, and which ones (if any) it doesn't belong to. Note that all four classes should be written for each graph.

(1)



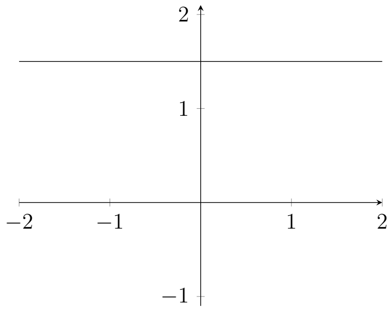
It belongs to polynomial, rational. Doesn't belong to linear, power.

(2)



It belongs to power, rational. Doesn't belong to linear, polynomial.

(3)



It belongs to linear polynomial, rational. Doesn't belong to power.