1. Quiz (10 mins)

2. Newton's method (15 mins)

How does the calculator find the roots, if it doesn't employ formulas? Let's look at the following picture: we have some polynomial, and we want to approximate the root. We start with an approximation x_0. How may we use tangent lines to get better approximations? (Draw picture)

Let's explain algebraically what happened here. We started with an approximation x_0, and we drew the tangent line at $(x_0, f(x_0)$. Its equation is $y - f(x_0) = f'(x_0)(x - x_0)$. The x-intercept is given by $(x_1, 0)$, and we can solve for x_1 if $f'(x_0) \neq 0$ given the equation of the line:

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$
$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

But we saw that we were able to get a better approximation by doing the same again. Given the new approximation, x_1, the point on the graph is $(x_1, f(x_1))$, then going through the same process again we get a new approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, if we keep iterating, given an approximation x_n we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Let's practice! **Exercise:** Estimate $\sqrt[6]{2}$ correct to eight decimal places.

Solution: $\sqrt[6]{2}$ is a solution of the equation $x^6 - 2$. So let $f(x) = x^6 - 2$, then $f'(x) = 6x^5$.

We know that $\sqrt[6]{2}$ is somewhere between 1 and 2, so let the first approximation be $x_0 = 1$. Since $f'(1) = 6$ is not close to 0, we may proceed. Then the general formula for the approximations will be

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$

So

$$x_1 = 1 - \frac{1^6 - 2}{6 \cdot 1^5} = 1 - \frac{-1}{6} = \frac{7}{6} \approx 1.16666667$$

$$x_2 = \frac{7}{6} - \frac{(\frac{7}{6})^6 - 2}{6 \cdot (\frac{7}{6})^5} \approx 1.12644368$$

If we keep doing so,

$$x_3 \approx 1.12249707$$
$$x_4 \approx 1.12246205$$
$$x_5 \approx 1.22246205$$

Since x_4 and x_5 agree to eight decimal places, a good approximation should be

$$\sqrt[6]{2} \approx 1.22246205.$$

(1.20)

3. Linear approximation (rest of time)

Let's get back to Newton's method: in the background of our pctures and calculations was the fact that for values of x close to a, the graph of $f(x)$ was looking like the tangent line at $(a, f(a))$, i.e. the tangent line was a good approximation for $f(x)$. (Draw picture)

The point-slope formula for a line is $y - y_0 = m(x - x_0)$. In this case, we want the tangent line at $(a, f(a))$, so the slope of the tangent line is the derivative $f'(a)$. So the tangent line has the equation $y - f(a) = f'(a)(x - a)$. Since we assume that the tangent line is a good approximation for the graph of $f(x)$, we will call

$$f(x) \approx f(a) + f'(a)(x - a)$$

the **linear approximation** or **tangent line approximation** of f at a.

Example: Let $f(x) = \sqrt{x+3}$, then $f'(x) = \frac{1}{2\sqrt{x+3}}$. Close to $a = 1$, the linear approximation is $f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{7}{4} + \frac{1}{4}x$. So we can approximate numbers like $\sqrt{3.98} = \sqrt{0.98+3}$ by

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$$

and $\sqrt{4.05} = \sqrt{1.05+3}$ by

$$\sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125.$$

What is this good for? In many cases in the physical world, we only look at some parameters that vary very little or are very small, in which case considering linear functions as approximations to more complicated functions makes our work much easier. One situation that you have already encountered was in physics class, in deriving a formula for the period of a pendulum (draw picture). Here we have $\theta(t)$ the angle at time t, g the gravitational acceleration, l the length of the pendulum, then the tangential acceleration $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta)$ is approximated by $\frac{d^2\theta}{dt^2} = -g\theta$ for small angles θ.

First question: why is this approximation ok? Well, at $a = 0$, the linear approximation of $f(x) = \sin(x)$ is $f(x) \approx f(0) + f'(0)(x-0) = \sin(0) + \cos(0)(x-0) = 0 + 1(x-0) = x$. Second question: where do we need the motion of the pendulum in real life, especially where θ is small? In fact, such approximations like $\sin(x) \approx x$ for small x or $\cos(x) = 1$ for small x are used often in modeling systems that are periodic, and periodic processes are found a lot in nature (sound waves, all other kinds of waves, life cycles in modeling populations, the economic cycle etc.) Of course, such approximations have their own shortcomings, and there is a lot of theory dealing with what to do when you can't necessarily approximate linearly; you can learn methods on how to deal with such situations in a differential equations class.

A cool example of an application is the following situation: I am on Earth, and a satellite is flying above the planet with velocity v. According to special relativity, me and the satellite will record two different times: satellite's clock recordts time T, my clock records time T_m. The equation relating the two times is

$$T_m = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where c is the speed of light. How are the two times different?

Let $x = \frac{v^2}{c^2}$, then $T_m = T \cdot \frac{1}{\sqrt{1-x}} = Tf(x)$, where $f(x) = \frac{1}{\sqrt{1-x}}$. Then the linear approximation of $f(x)$ at $x = 0$ is $f(x) \approx f(0) + f'(0)(x-0) = 1 + \frac{1}{2}x$ since $f'(x) = \frac{-1}{2}(1-x)^{-3/2}(-1) = \frac{1}{2}(1-x)^{-3/2}$ and $f'(0) = \frac{1}{2}1^{-3/2} = \frac{1}{2}$. Then

$$T_m \approx T(1 + \frac{1}{2}\frac{v^2}{c^2}) \approx T,$$

since the speed of light is much larger than the speed of the satelite, so $\frac{v^2}{c^2} \approx 0$.

Such models have actually been used by engineers to calibrate the transmitters on the GPS satelites; since there will be a difference in the time on our GPS devices and the satellites, they needed to make sure there won't be a significant error in the devices's estimate of our position.

Group work:

1. Use Newton's method to find x_2, the third approximation to the root of $2x^3 - 3x^2 + 2 = 0$, given $x_0 = -1$. Give your answer to 4 decimal places.

Solution: First, given $f(x) = 2x^3 - 3x^2 + 2$, we find $f'(x) = 6x^2 - 6x$. Then

$$x_1 = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{-3}{12} = -1 + \frac{1}{4} = \frac{-3}{4}$$

and

$$x_2 = -0.75 - \frac{f(-0.75)}{f'(-0.75)} \approx -0.6825$$

2. Use linear approximation to approximate the following functions:

$$f(x) = \sin(x) \text{ at } a = \pi/6$$

and

$$g(x) = e^x \text{ at } a = 0$$

Solution: $f'(x) = \cos(x)$, and at $a = \frac{\pi}{6}$,

$$f(x) \approx \sin(\pi/6) + \cos(\pi/6)(x - \pi/6) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6})$$

$g'(x) = e^x$, and at $a = 0$, we have

$$g(x) \approx e^0 + e^0(x - 0) = 1 + x$$

3. Once you found the approximation of $g(x) = e^x$ above, approximate $\sqrt[n]{e} = e^{1/n}$ for a natural number n. Taking nth powers of both sides, what approximation of e do you get? Since the approximation of e^x gets bettwe as x gets closer to 0, the approximation of e above

gets better as n grows larger (since then $1/n$ gets closer to 0). Taking limits as $n \to \infty$, you should get that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

The constant e in this form has been discovered in 1683 by Jacob Bernoulli while studying a question about compound interest.

Solution: Since above we saw that $e^x \approx 1 + x$ when x is small, $e^{1/n} \approx 1 + 1/n$, so taking nth powers of both sides, we have $e \approx (1 + 1/n)^n$.