DIFFERENTIATION WORKSHEET II

Exercise 1. Find the following derivatives:

(1) $(\sin(x)\cos(x))' = (\sin(x))'\cos(x) + \sin(x)(\cos(x))' = \cos^2(x) - \sin^2(x)$

(2) $((2x^2 + e^x)\tan(x))' = (2x^2 + e^x)'\tan(x) + (2x^2 + e^x)(\tan(x))' = (4x + e^x)\tan(x) + (2x^2 + e^x)\sec^2(x)$

(3) $\left(\frac{\cos(x)}{1-\sin(x)}\right)' = \frac{(\cos(x))'(1-\sin(x)) - \cos(x)(1-\sin(x))'}{(1-\sin(x))^2} =$

$$\frac{-\sin(x)(1-\sin(x)) - \cos(x)(-\cos(x))}{(1-\sin(x))^2} = \frac{-\sin(x) + \sin^2(x) + \cos^2(x)}{(1-\sin(x))^2} = \frac{1-\sin(x)}{(1-\sin(x))^2} = \frac{1}{1-\sin(x)}$$

(4) $\left(\frac{1+\cos(x)}{x}\right)' = \frac{(1+\cos(x))'x - (1+\cos(x))(x)'}{x^2} = \frac{-\sin(x)x - (1+\cos(x))}{x^2}$

Exercise 2. What is the 17th derivative of $\sin(x)$?

Solution: By our formulas for the derivatives of $\sin(x), \cos(x)$, we have

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$
$$\frac{d^2}{dx^2}(\sin(x)) = -\sin(x)$$
$$\frac{d^3}{dx^3}(\sin(x)) = -\cos(x),$$
$$\frac{d^4}{dx^4}(\sin(x)) = \sin(x),$$

so the higher derivatives of $\sin(x)$ repeat themselves after 4 iterations. Therefore,

$$\frac{d^8}{dx^8}(\sin(x)) = \sin(x),$$
$$\frac{d^{12}}{dx^{12}}(\sin(x)) = \sin(x),$$
$$\frac{d^{16}}{dx^{16}}(\sin(x)) = \sin(x),$$

and therefore

$$\frac{d^{17}}{dx^{17}}(\sin(x)) = \cos(x),$$

Exercise 3: Using the limit definition of the derivative, show that the sum rule is true, that is, that

$$(f(x) + g(x))' = f'(x) + g'(x)$$

Solution:

$$(f(x)+g(x))' = \lim_{h\to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

The first equality follows from the definition of the derivative. The second from rearranging the terms in the numerator. Then the whole expression is equal to

$$\lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=f'(x)+g'(x)$$

The first equality above is from rewriting the fraction as the sum of two fractions. The second equality follows since each of the two limits is the definition of a derivative.

Exercise 4: Find the following derivatives:

(1) $(x^2\cos(x)e^x)' = (x^2\cos(x))'e^x + x^2\cos(x)(e^x)' = ((x^2)'\cos(x) + x^2(\cos(x))')e^x + x^2\cos(x)e^x = (2x\cos(x) - x^2\sin(x))e^x + x^2\cos(x)e^x$

(2) $\left(\frac{x^2\cos(x)}{e^x}\right)' = \frac{(x^2\cos(x))'e^x - x^2\cos(x)(e^x)'}{(e^x)^2} = \frac{(2x\cos(x)-x^2\sin(x))e^x - x^2\cos(x)e^x}{(e^x)^2} = \frac{2x\cos(x)-x^2\sin(x)-x^2\cos(x)}{e^x}$

The second equality follows since in part (1) we found $(x^2\cos(x))'e^x = (2x\cos(x) - x^2\sin(x))e^x$