Chain Rule Practice

Compute the following derivatives.

1. Find the derivative of
$$f(x) = e^{\cos(x)}.$$

   Answer:

   We observe that $f(x)$ can be decomposed into $g(h(x))$ where $g(x) = e^x$ and $h(x) = \cos(x)$, since we must first take the cosine of x and then raise e to that power.

   Thus by the chain rule
$$f'(x) = g'(h(x))h'^{(x)} = e^{h(x)}h'(x) = e^{\cos(x)}h'(x) = e^{\cos(x)}(-\sin(x)) = -e^{\cos(x)}\sin(x).$$

2. Find the derivative of
$$g(x) = \cos(x^2)\sin(x^3).$$

   Answer:

   We observe that $g(x)$ can be broken into a non-trivial product $g(x) = g_1(x)g_2(x)$ where $g_1(x) = \cos(x^2)$, and $g_2(x) = \sin(x^3)$.

   Thus we need to compute the derivative of $g_1(x)$ and $g_2(x)$. We observe that g_1 can be decomposed as $f_1(h_1(x))$ where $f_1(x) = \cos(x)$ and $h_1(x) = x^2$. Thus by the chain rule
$$g_1'(x) = f_1'(h_1(x))h_1'(x) = -\sin(h_1(x))h_1'(x) = -\sin(x^2)h_1'(x) = -\sin(x^2)2x = -2x\sin(x^2).$$

   We also observe that g_2 can be decomposed as $f_2(h_2(x))$ where $f_2(x) = \sin(x)$ and $h_2(x) = x^3$. Thus by the chain rule
$$g_2'(x) = f_2'(h_2(x))h_2'(x) = \cos(h_2(x))h_2'(x) = \cos(x^3)h_2'(x) = \cos(x^3)3x^2 = 3x^2\cos(x^3).$$

   Thus by the product rule
$$g'(x) = g_1(x)g_2'(x) + g_1'(x)g_2(x) = \cos(x^2)\left(2x^2\cos(x^3)\right) + \sin(x^3)\left(-3x\sin(x^2)\right),$$

   so
$$g'(x) = 3x^2\cos(x^2)\cos(x^3) - 2x\sin(x^3)\sin(x^2)$$

3. Find the derivative of

$$\ell(x) = \cos(xe^x).$$

Answer:

We observe that $\ell(x)$ can be decomposed into $f(g(x))$ where $g(x) = xe^x$ and $f(x) = \cos(x)$. Thus by the chain rule

$$\ell'(x) = f'(g(x))g'^{(x)} = -\sin(g(x))g'(x) = -\sin(xe^x)g'(x) = -\sin(xe^x)\frac{d}{dx}(g(x))$$

Since $g(x)$ can be written as the product of two non-trivial functions $g_1 g_2$ where $g_1(x) = x$ and $g_2(x) = e^x$. Thus, by the product rule, we have

$$g'(x) = g_1'(x)g_2(x) + g_2'(x)g_1(x) = 1e^x + xe^x = (1+x)e^x$$

Thus

$$\ell'(x) = -\sin(xe^x)(1+x)e^x$$

4. Find the derivative of

$$k(x) = 2^{\left(3^{(5^x)}\right)}.$$

Answer:

We observe that $k(x)$ can be written as the composition of three functions $f(g(h(x)))$ where $f(x) = 2^x$, $g(x) = 3^x$, and $h(x) = 5^x$.

$$\frac{d}{dx}k(x) = f'(g(h(x)))\frac{d}{dx}g(h(x)) = \ln(2)2^{\left(3^{(5^x)}\right)}\frac{d}{dx}\left(3^{(5^x)}\right).$$

Thus we need to apply the chain rule to the composition of g and h. Observe that

$$\frac{d}{dx}g(h(x)) = g'(h(x))\frac{d}{dx}h(x) = \ln(3)3^{(5^x)}\ln(5)5^x.$$

Thus

$$\frac{d}{dx}k(x) = \ln(2)2^{\left(3^{(5^x)}\right)}\ln(3)3^{(5^x)}\ln(5)5^x = \ln(2)\ln(3)\ln(5)2^{\left(3^{(5^x)}\right)}3^{(5^x)}5^x$$