Math 1 Lecture 26

Dartmouth College

Wednesday 11-09-16

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Reminders/Announcements

- Last written HW due today!
- WebWork due Friday (Taylor Polys) and Monday (Misc Review)
- All written work will be returned by the end of this week and thus all grades except for the final will be available Monday
- Final Exam Review during x-hour, Friday, and Monday (Do we want another review during the reading period?)
- Look for an email later today with information regarding the final exam
- I'm putting together a comprehensive review document which will be available at the following link: `https://math.dartmouth.edu/~m1f16/MATH1Docs/Musty-Lecture-27-Slides-11-11-Fri.pdf`

This Week

- Linear Approximations (Monday)
- Taylor Polynomials (Wednesday)
- Exam Review (Thursday, Friday, following Monday, and possibly more during reading period)

Taylor Polynomials

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Given a function f differentiable in an interval containing the real number a, we define the **degree n Taylor polynomial of f centered at a** as follows:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

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Notice that the linearization of f at a is simply $T_1(x)$.

Example

Let $f(x) = \sin(x)$ and $a = 0$. Find $T_7(x)$.

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Solution:

$$\begin{aligned} T_7(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(7)}(0)}{7!}x^7 \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}. \end{aligned}$$

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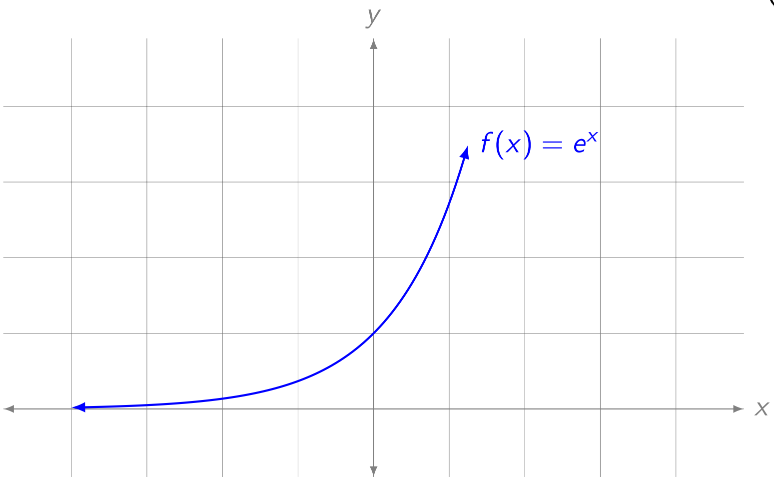
Solution:

$$T_6(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(6)}(0)}{6!}x^6$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

Taylor Polynomial Approximations



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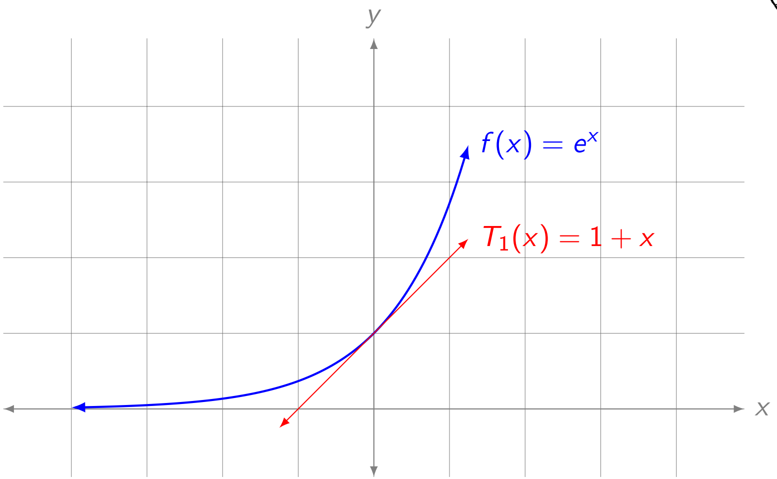

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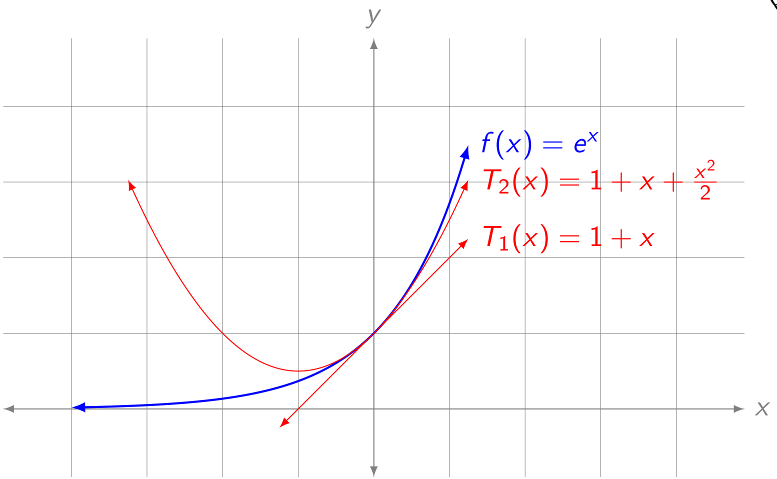

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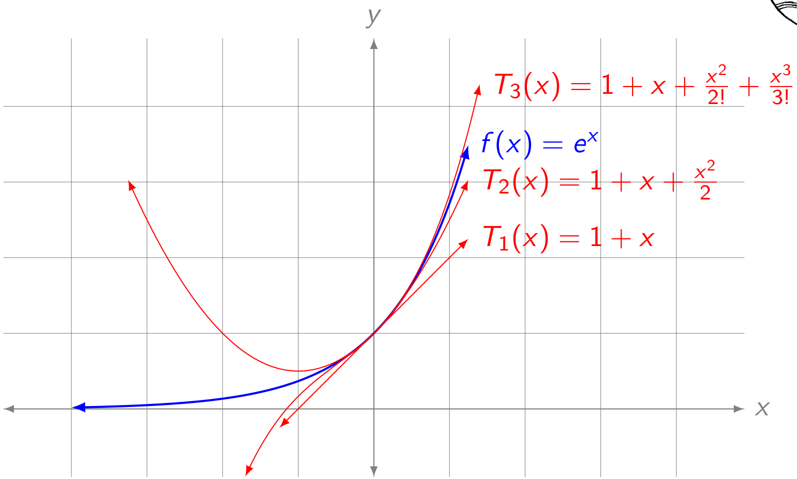

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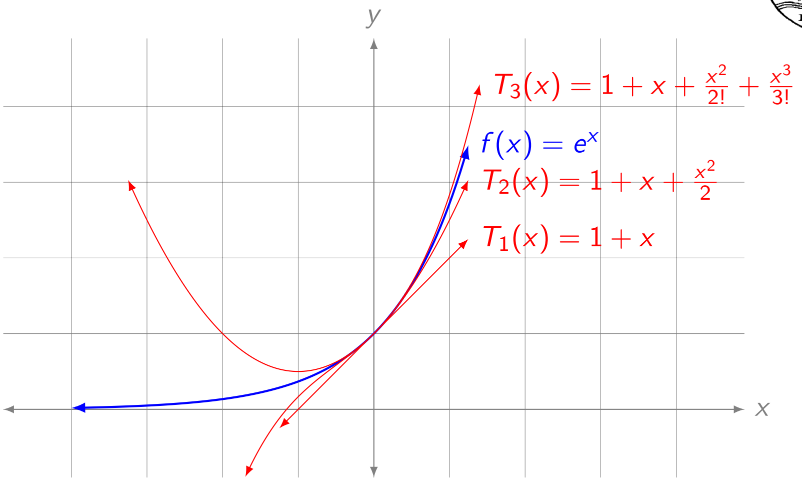

$$e^x = \lim_{n\to\infty} T_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \quad \text{for all } x$$

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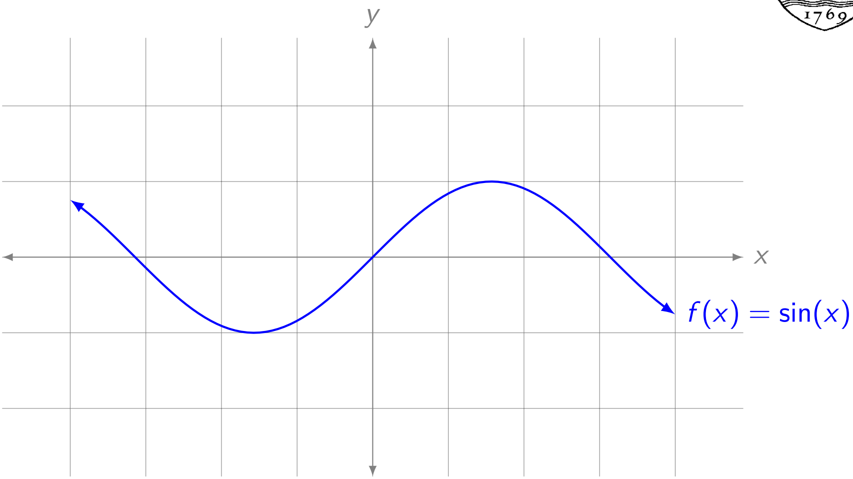

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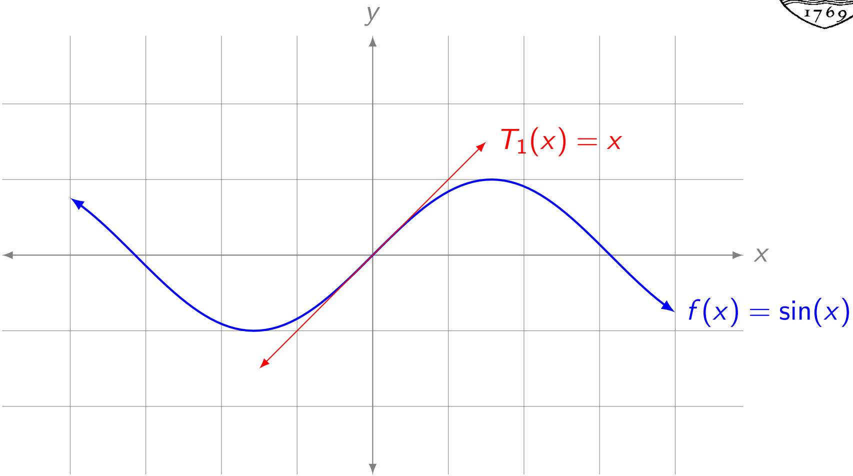

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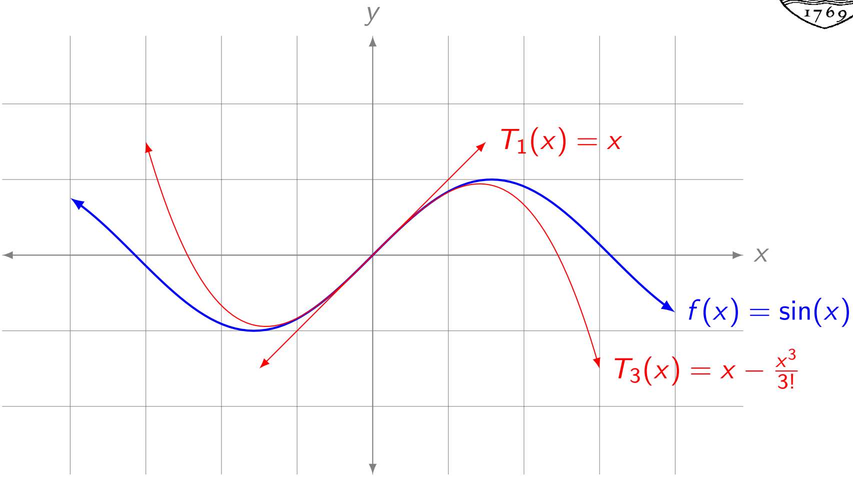

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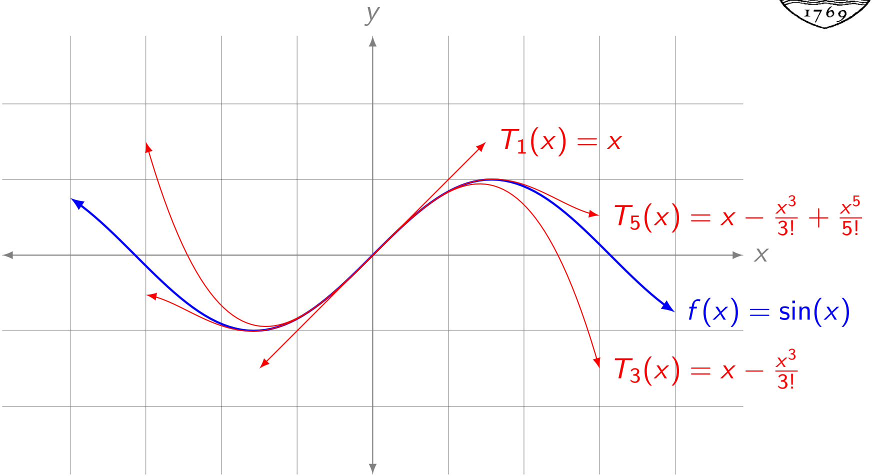

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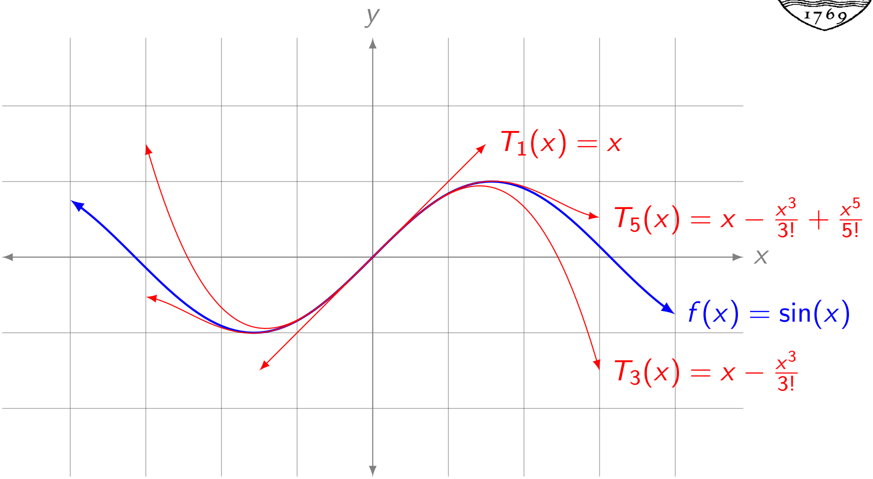

$$\sin(x) = \lim_{n \to \infty} T_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

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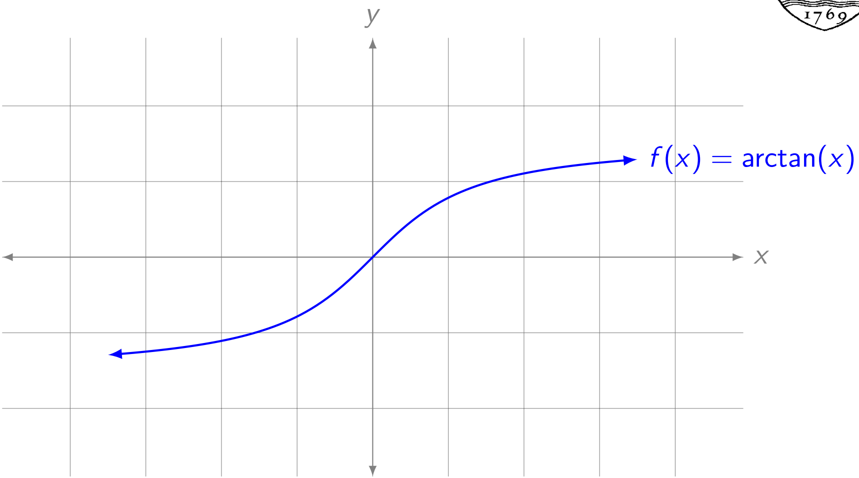

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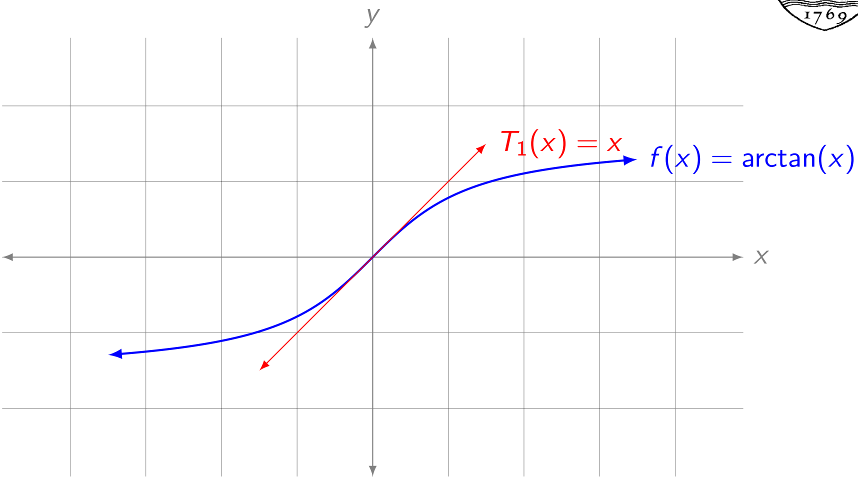

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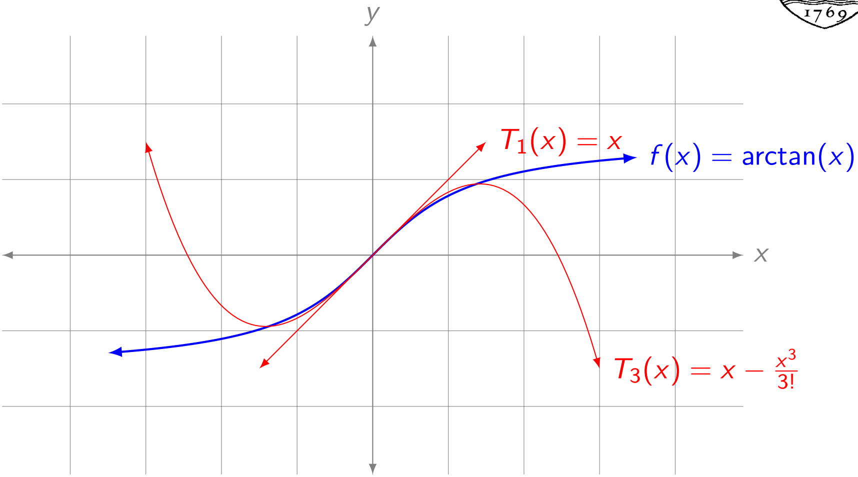

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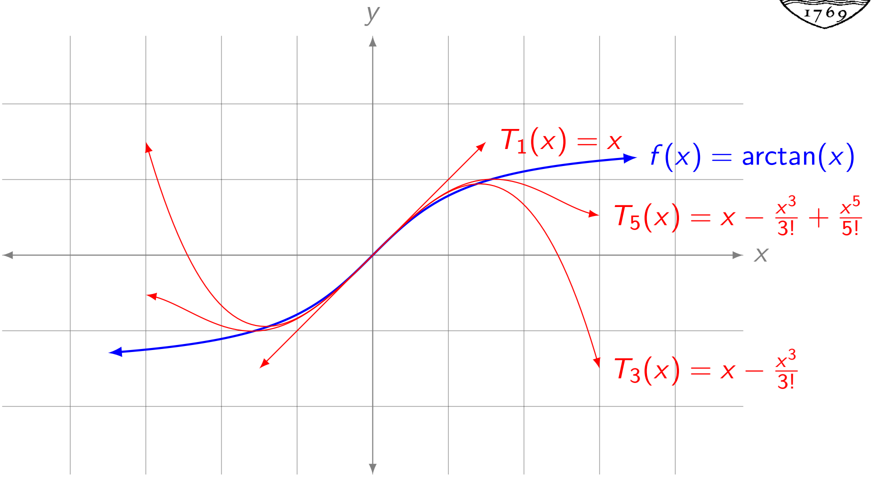

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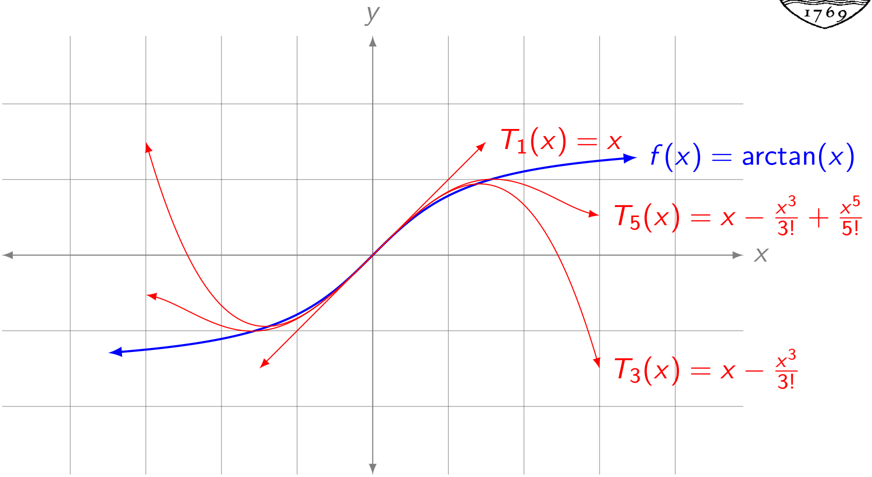

$$\arctan(x) = \lim_{n\to\infty} T_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for } -1 < x < 1$$

$\frac{d}{dx}(\sin(x)) = \cos(x)$

Suppose we follow the suggestive pattern from the previous slides and write

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$
$$\cos(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

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Then

$$\begin{aligned}
\frac{d}{dx}(\sin(x)) &\approx \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) \\
&\approx 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \dots \\
&\approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots
\end{aligned}$$

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and all of this can be made precise using **infinite series**.

Examish Exercises

1. For the following assume $a = 0$ and find $T_3(x)$:
   (a) $f(x) = \frac{1}{1-x}$
   (b) $f(x) = e^x$
   (c) $f(x) = \ln(1 + x)$
   (d) $f(x) = xe^x$
2. Find $T_4(x)$ for the following:
   (a) $f(x) = x^5 + 2x^3 + x$, $a = 2$
   (b) $f(x) = e^{2x}$, $a = 3$
   (c) $f(x) = \sin(x)$, $a = \pi$
   (d) $f(x) = \sqrt{x}$, $a = 16$