Math 1 Lecture 24

Dartmouth College

Friday 11-04-16

Contents

Reminders/Announcements

- Last Quiz Monday!
- Last written HW due Wednesday!
- WebWork due Monday. . . but not the last one

A Week in the Life

- Implicit differentiation
- $(f^{-1})'(a) = 1/f'(f^{-1}(a))$
- Derivatives of inverse trigonometric functions
- Derivatives of logarithmic and exponential functions
- Logarithmic differentiation
- Newton's Method (today)

Newton's Method

Given a function $f(x)$ and a starting value x_0, **Newton's method** produces a sequence of real numbers

$$x_0, x_1, x_2, \ldots$$

such that (in most cases…but not always)

$$\lim_{n \to \infty} x_n = \zeta$$

where ζ satisfies $f(\zeta) = 0$.

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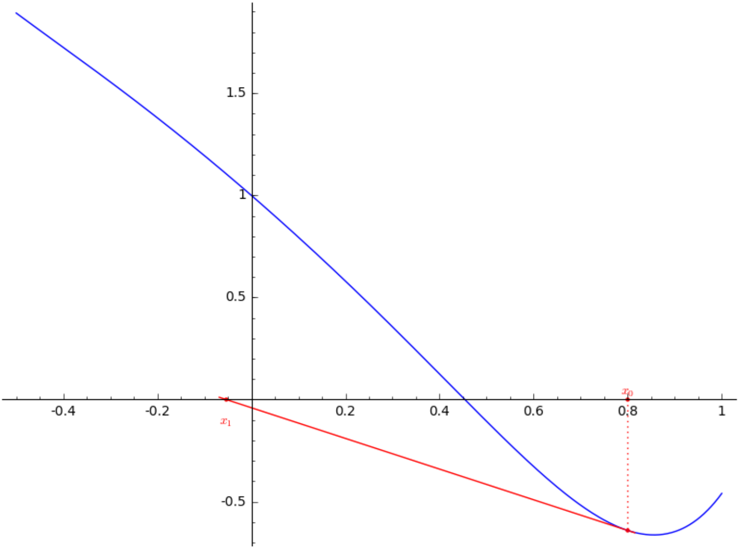
where ζ satisfies $f(\zeta) = 0$. That is, Newton's method produces a sequence that converges to a **root** of f.
The x_n are computed inductively by the formula:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}.$$

Newton's Method



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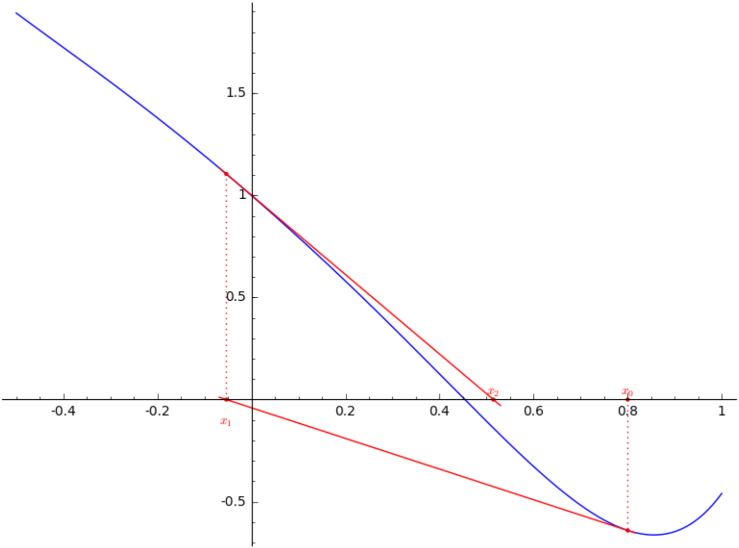

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f has a root $\zeta \in [0, 1]$. How do we know this? Since f is continuous and

$$f(0) = 0 - 0 + \cos(0) = 1 > 0$$
$$f(1) = 1 - 2 + \cos(1) = -1 + \cos(1) < 0,$$

by the intermediate value theorem (oh yeah that thing), f has a root (call it ζ) in the interval $[0, 1]$.

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We can approximate ζ using Newton's method...

Newton's Method

Input:

- $f(x) = x^6 - 2x + \cos(x)$
- $x_0 = 0.8$

Output:

- a sequence $x_0, x_1, x_2, \ldots$

Newton's Method

Setup:

- $f(x) = x^6 - 2x + \cos(x), \quad f'(x) = 6x^5 - 2 - \sin(x)$
- $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_0 = 0.8$

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Compute x_1:

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= x_0 - \frac{x_0^6 - 2x_0 + \cos(x_0)}{6x_0^5 - 2 - \sin(x_0)} \\ &= 0.8 - \frac{(0.8)^6 - 2(0.8) + \cos(0.8)}{6(0.8)^5 - 2 - \sin(0.8)} \\ &= -0.05341367630263 5\ldots \end{aligned}$$

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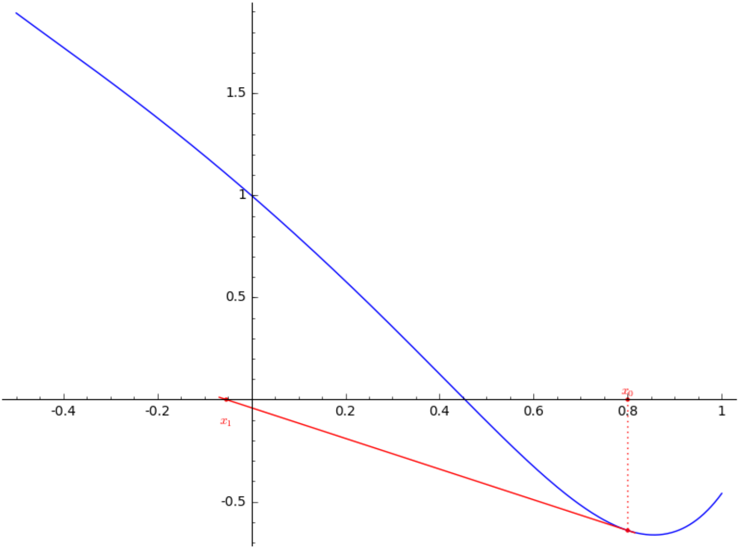
Compute x_2:

$$\begin{aligned}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&= x_1 - \frac{x_1^6 - 2x_1 + \cos(x_1)}{6x_1^5 - 2 - \sin(x_1)} \\
&= (-0.05\ldots) - \frac{(-0.05\ldots)^6 - 2(-0.05\ldots) + \cos(-0.05\ldots)}{6(-0.05\ldots)^5 - 2 - \sin(-0.05\ldots)} \\
&= 0.51444467609636\ldots
\end{aligned}$$

Newton's Method



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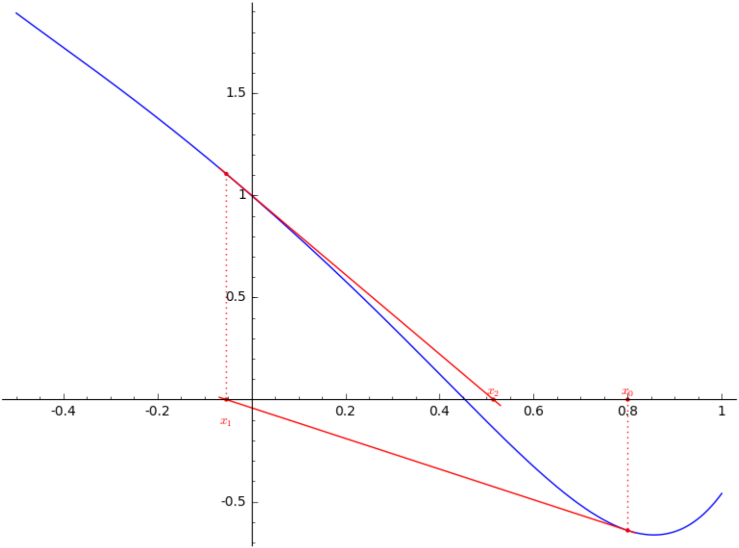

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Output:

$$\begin{aligned}
x_0 &= 0.8 \\
x_1 &= -0.053413676302635\ldots \\
x_2 &= 0.514444676096366 62\ldots \\
x_3 &= 0.4530231701878661 7\ldots \\
x_4 &= 0.4537660371612551 8\ldots && \text{3 decimal places} \\
x_5 &= 0.4537660807937113 4\ldots && \text{7 decimal places} \\
x_6 &= 0.4537660807937114 9\ldots && \text{15 decimal places}
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$$x_4 = 0.4537660371612551 8\ldots \qquad \text{3 decimal places}$$
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https://goo.gl/KLQPjK

Practice Quiz (10 minutes)

1. Consider the implicit function

$$y^4 - 53 = \frac{1}{2}x^3 + 5x.$$

   (a) What is $\frac{dy}{dx}$?
   (b) What is the slope of the tangent line at the point $x = -4$, $y = 1$?

2. Consider the function

$$f(x) = \ln(x - 3).$$

   (a) What is $f'(x)$?
   (b) What is the domain of $f'(x)$ as an interval?

Examish Exercises

1. Let $f(x) = x^7 + 4$ and $x_0 = -1$. Compute x_1 and x_2 using Newton's method.
   Solution:
   `https://goo.gl/V2kOyU`
2. Approximate $\sqrt[4]{75}$ to eight decimal places using Newton's method.
   Solution:
   `https://goo.gl/Te5YjK`

Examish (Mon)

1. Find $\frac{dy}{dx}$ for the equation $y^2 = x^3 - x$.
2. Find $\frac{dy}{dx}$ for the equation $x^3 + y^3 = 6xy$.
3. Find y' for the equation $\sin(x + y) = y^2 \cos(x)$.
4. Find the equation of the tangent line to the curve $x^2 - xy - y^2 = 1$ at the point $(2, 1)$.
5. Find the equation of the tangent line to the curve $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at the point $(0, 1/2)$.
6. Find the equation of the tangent line to the curve $x^{2/3} + y^{2/3} = 4$ at the point $\left(-3\sqrt{3}, 1\right)$.

Examish (Mon)

1. Let $f(x) = 3x^3 + 4x^2 + 6x + 5$ and $a = 5$. Find $(f^{-1})'(a)$.
2. Let $f(x) = x^3 + 3\sin(x) + 2\cos(x)$ and $a = 2$. Find $(f^{-1})'(a)$.
3. Let $f(x) = \sqrt{x^3 + 4x + 4}$ and $a = 3$. Find $(f^{-1})'(a)$.
4. Suppose f^{-1} is the inverse function of a differentiable function f and $f(4) = 5$, $f'(4) = 2/3$. Find $(f^{-1})'(5)$.

Examish (Mon)

1. Find the derivative of $y = x \arcsin(x) + \sqrt{1 - x^2}$.
2. Find the derivative of $y = \arctan \sqrt{\frac{1-x}{1+x}}$.
3. Find the derivative of $f(\theta) = \arctan(\cos(\theta))$.
4. Find y' if $\arctan(x^2 y) = x + xy^2$.

Examish (Wed)

Find the derivative of each function and the domain on which it is valid.

1. $y = \ln(x + 5)$
2. $y = \ln|x + 5|$

Examish (Wed)

1. $f(x) = x \ln x - x$
2. $f(x) = \sin(\ln x)$
3. $y = \ln \frac{1}{x}$
4. $g(x) = \ln(xe^{-2x})$
5. $f(x) = \log_{10} x$
6. $h(x) = \log_{10} \sqrt{x}$
7. $y = 2^x$
8. $y = 5^{2x+1}$
9. $y = (x^2 + 2)^2 (x^4 + 4)^4$
10. $y = (2x + 1)^x$