Math 1 Lecture 22

Dartmouth College

Monday 10-31-16

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- WebWork due Wednesday
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- Written HW due Wednesday

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- Chain Rule

Implicitly defined functions

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What does the solution set of this equation look like? A circle! How about $y^2 = x^3 - x$?

Implicit Differentiation

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The resulting expression for y' might end up being in terms of x and y. When y' is evaluated at a point (x, y), the resulting number represents the slope of the tangent line to the implicitly defined curve at the point (x, y).

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For the first term on the LHS we have that $\frac{d}{dx}(x^2) = 2x$, but what about the term $\frac{d}{dx}(y^2)$?

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$$\frac{dy}{dx} = y' = -\frac{2x}{2y} = -\frac{x}{y}.$$

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$$y - 4 = -\frac{3}{4}(x - 3).$$

Derivatives of Inverse Functions

If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

Let $f(x) = 2x + \cos(x)$. Find $(f^{-1})'(1)$.

Solution:
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$$\begin{aligned}
(f^{-1})'(1) &= \frac{1}{f'(f^{-1}(1))} \\
&= \frac{1}{f'(0)} \\
&= \frac{1}{2 - \sin(0)} \\
&= \frac{1}{2 - 0} \\
&= \frac{1}{2}.
\end{aligned}$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad -1 < x < 1$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \quad -1 < x < 1$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Let $y = \arctan(x^2)$. Find y'.

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Solution:

$$y' = \frac{1}{1 + \left(x^2\right)^2} \cdot (x^2)'$$
$$= \frac{2x}{1 + x^4}.$$

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If we also want to find the equation of the tangent line at the point $(1, \pi)$, then we simply evaluate $y'(1) = -\sqrt{3}$. Then the tangent line is defined by the equation

$$y - \pi = -\sqrt{3}(x-1).$$

Examish

1. Find $\frac{dy}{dx}$ for the equation $y^2 = x^3 - x$.
2. Find $\frac{dy}{dx}$ for the equation $x^3 + y^3 = 6xy$.
3. Find y' for the equation $\sin(x + y) = y^2 \cos(x)$.
4. Find the equation of the tangent line to the curve $x^2 - xy - y^2 = 1$ at the point $(2, 1)$.
5. Find the equation of the tangent line to the curve $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at the point $(0, 1/2)$.
6. Find the equation of the tangent line to the curve $x^{2/3} + y^{2/3} = 4$ at the point $\left(-3\sqrt{3}, 1\right)$.

Examish

1. Let $f(x) = 3x^3 + 4x^2 + 6x + 5$ and $a = 5$. Find $(f^{-1})'(a)$.
2. Let $f(x) = x^3 + 3\sin(x) + 2\cos(x)$ and $a = 2$. Find $(f^{-1})'(a)$.
3. Let $f(x) = \sqrt{x^3 + 4x + 4}$ and $a = 3$. Find $(f^{-1})'(a)$.
4. Suppose f^{-1} is the inverse function of a differentiable function f and $f(4) = 5$, $f'(4) = 2/3$. Find $(f^{-1})'(5)$.

Examish

1. Find the derivative of $y = x \arcsin(x) + \sqrt{1 - x^2}$.
2. Find the derivative of $y = \arctan \sqrt{\frac{1-x}{1+x}}$.
3. Find the derivative of $f(\theta) = \arctan(\cos(\theta))$.
4. Find y' if $\arctan(x^2 y) = x + xy^2$.