Math 1 Lecture 15

Dartmouth College

Friday 10-14-16

Contents

Reminders/Announcements

- WebWork due Monday and Wednesday next week
- Written Homework due Wednesday
- Exam#2 is Thursday 10/20/16 and will cover material from Trigonometry up to and NOT including derivatives
- Exam review during x-hour 10/20/16
- Exam Review Slides: `https://math.dartmouth.edu/~m1f16/MATH1Docs/Musty-x-hour-Slides-10-13-Thur.pdf`
- Because of the exam there will be no WebWork due Friday 10/21/16

Squeeze Theorem

Suppose the following two conditions are satisfied:

- $f(x) \leq g(x) \leq h(x)$ for all x in an open interval containing a except possibly at a.
-

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$$-x^2 \leq x^2 \sin(1/x) \leq x^2.$$

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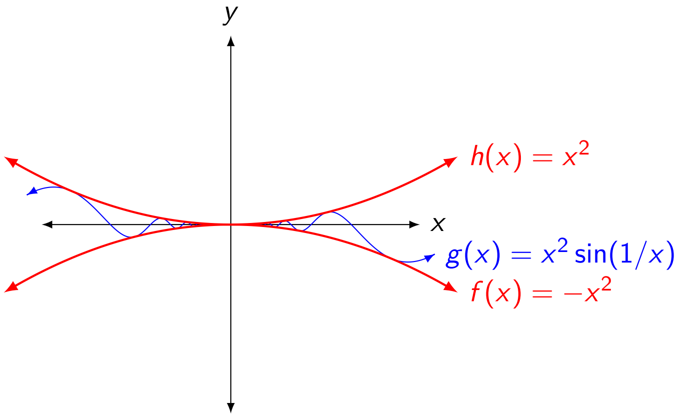
Can you see how to apply the squeeze theorem now? Take $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$... A picture illustrates what is going on here...



y
x
h(x) = x^2
g(x) = x^2 sin(1/x)
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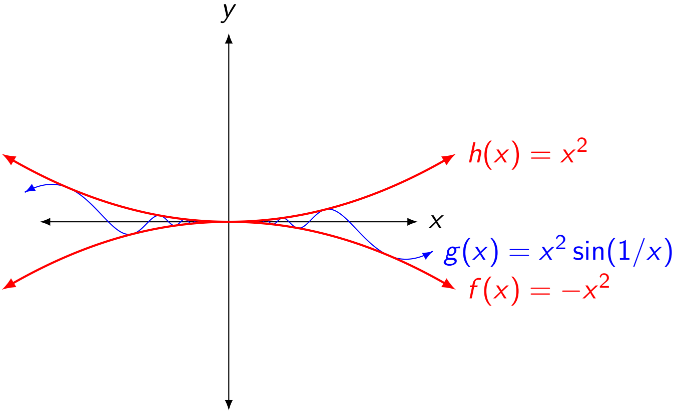

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Solution:

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Growth Rates of Functions

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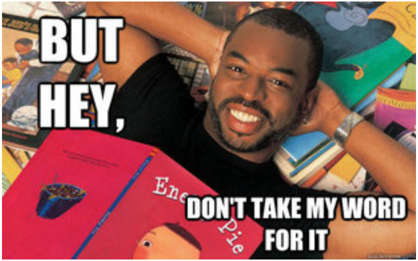

Let's talk about

$$\lim_{x \to \infty} \frac{3\sqrt{x}}{\sqrt{x} + 5}.$$

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Solution:
The limit is 3.

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Solution:

The limit is ∞. Divide numerator and denominator by $x^{3/2}$ to see this algebraically.

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$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x^3+4x}}.$$

Solution:

The limit is ∞. Divide numerator and denominator by $x^{3/2}$ to see this algebraically. Similarly,

$$\lim_{x\to\infty} \frac{\sqrt{x^3+4x}}{x^2} = 0.$$

Let's talk about

$$\lim_{x \to \infty} \frac{(\ln x)^{20}}{x}.$$

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Solution:
The limit is 0. Why? Because logarithms grow slower than polynomials.

Let's talk about

$$\lim_{x\to\infty} \frac{\pi \cdot 2^x + 50x^2 + 5}{51x^2 - 6 + \sqrt{3} \cdot 2^x}.$$

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$$\lim_{x\to-\infty} \frac{\pi \cdot 2^x + 50x^2 + 5}{51x^2 - 6 + \sqrt{3} \cdot 2^x}.$$

Solution:
Now $2^x \to 0$. See why? Thus the limit is $50/51$. Why? As $x \to -\infty$ the quadratic terms are growing the fastest.

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Solution:

$$\begin{aligned}\lim_{x\to\infty} \arctan\left(\frac{x^2}{x^2+4}\right) &= \arctan\left(\lim_{x\to\infty} \frac{x^2}{x^2+5}\right)\\ &= \arctan(1)\\ &= \pi/4.\end{aligned}$$

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$$\begin{aligned}\lim_{x\to\infty} \arctan\left(\frac{x^2}{4-x^2}\right) &= \arctan\left(\lim_{x\to\infty} \frac{x^2}{4-x^2}\right)\\ &= \arctan(-1)\\ &= -\pi/4.\end{aligned}$$

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Solution:

As $x \to \infty$ the expression under the radical goes to $1\ldots$ So this expression is approaching arctan(1)...which is? $\pi/4$.

Compute

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Solution:

The limit is 1. To see this, multiply numerator and denominator by $\sqrt{1+x} + \sqrt{1-x}$ and simplify.

Compute

$$\lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}.$$

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$$\lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}.$$

Solution:

Simplify the fractions to cancel h in the denominator limit is $-2/x^3$.

Let

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}.$$

For what values of the constant c is the function f continuous on $(-\infty, \infty)$?

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Solution:
Since both defining functions are continuous (on all of $\mathbb{R}$), the only place we need to check for continuity is at 2. By the defining expressions we see that continuity at 2 depends only on the equality

$$c \cdot 2^2 + 2 \cdot 2 = 2^3 - c \cdot 2.$$

Since this equality is satisfied only by $c = 2/3$, then this is the unique value of c making f continuous everywhere.