Math 1 Lecture 13

Dartmouth College

Monday 10-10-16

Contents

Reminders/Announcements

The Definition of $\lim_{x \to a} f(x)$

Properties of Limits

Examples of Limits

Reminders/Announcements

- WebWork due Wednesday
- Written Homework due Wednesday
- Quiz Today. . . like now

Mathematical definition of $\lim_{x \to a} f(x)$

Let f be defined on some interval containing the real number a except possibly at a itself.

Mathematical definition of $\lim_{x \to a} f(x)$

Let f be defined on some interval containing the real number a except possibly at a itself.

Then we say "the limit of $f(x)$ as x approaches a is L, and we write $\lim_{x \to a} f(x) = L$" if the following holds:

Mathematical definition of $\lim_{x \to a} f(x)$

Let f be defined on some interval containing the real number a except possibly at a itself.

Then we say "the limit of $f(x)$ as x approaches a is L, and we write $\lim_{x \to a} f(x) = L$" if the following holds:

For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Mathematical definition of $\lim_{x \to a} f(x)$

Let f be defined on some interval containing the real number a except possibly at a itself.

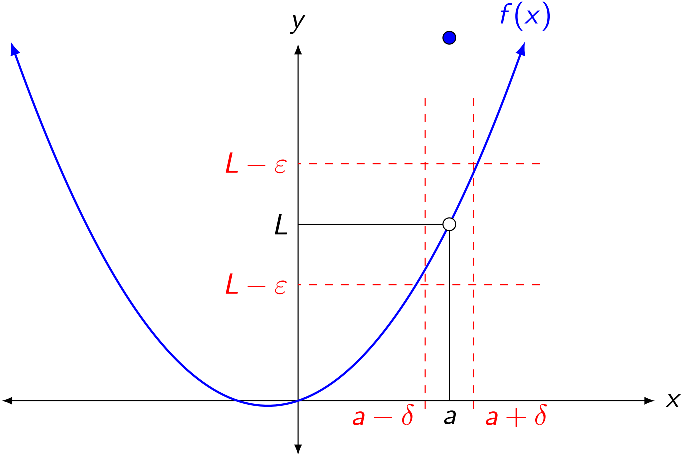
Then we say "the limit of $f(x)$ as x approaches a is L, and we write $\lim_{x \to a} f(x) = L$" if the following holds:

For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

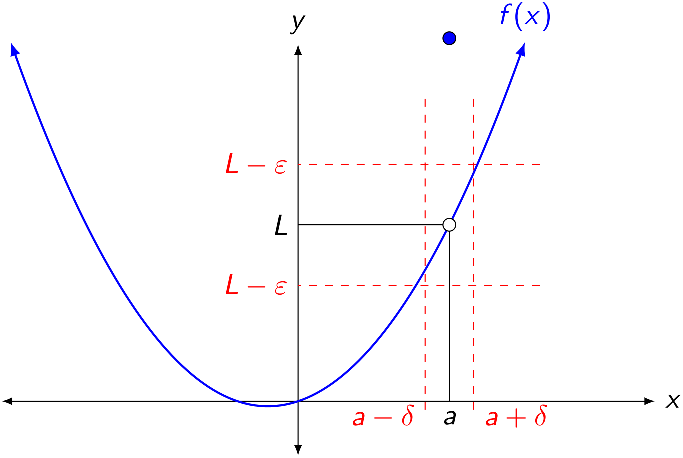
Show me a picture!

Definition. . . with a picture. . . fixed the typo!



Note that $a \in \mathbb{R}$ is constant.

Definition... with a picture... fixed the typo!😉



Note that $a \in \mathbb{R}$ is constant. Given $\varepsilon > 0$ we want δ so that

$$a-\delta < x < a+\delta \implies L-\varepsilon < f(x) < L+\varepsilon.$$

Prove that $\lim_{x\to 1}(x+2)=3$

Prove that $\lim_{x \to 1}(x+2) = 3$

First note that $a = 1$ and $L = 3$ and let $\varepsilon > 0$ be given.

Prove that $\lim_{x \to 1}(x + 2) = 3$

First note that $a = 1$ and $L = 3$ and let $\varepsilon > 0$ be given. Now we observe that

Prove that $\lim_{x\to 1}(x+2) = 3$

First note that $a = 1$ and $L = 3$ and let $\varepsilon > 0$ be given. Now we observe that

$$\begin{aligned} |f(x) - L| < \varepsilon &\implies 3 - \varepsilon < x + 2 < 3 + \varepsilon \\ &\implies 1 - \varepsilon < x < 1 + \varepsilon \\ &\implies |x - 1| < \varepsilon. \end{aligned}$$

Prove that $\lim_{x\to 1}(x+2) = 3$

First note that $a = 1$ and $L = 3$ and let $\varepsilon > 0$ be given. Now we observe that

$$\begin{aligned} |f(x) - L| < \varepsilon &\implies 3 - \varepsilon < x + 2 < 3 + \varepsilon \\ &\implies 1 - \varepsilon < x < 1 + \varepsilon \\ &\implies |x - 1| < \varepsilon. \end{aligned}$$

So if we take $\delta = \varepsilon$, then we get

Prove that $\lim_{x\to 1}(x+2) = 3$

First note that $a = 1$ and $L = 3$ and let $\varepsilon > 0$ be given. Now we observe that

$$\begin{aligned} |f(x) - L| < \varepsilon &\implies 3 - \varepsilon < x + 2 < 3 + \varepsilon \\ &\implies 1 - \varepsilon < x < 1 + \varepsilon \\ &\implies |x - 1| < \varepsilon. \end{aligned}$$

So if we take $\delta = \varepsilon$, then we get

$$\begin{aligned} |x - 1| < \delta &\implies -\delta < x - 1 < \delta \\ &\implies 1 - \delta < x < 1 + \delta \\ &\implies 3 - \delta < x + 2 < 3 + \delta \\ &\implies |\underbrace{x + 2}_{f(x)} - 3| < \delta \\ &\implies |f(x) - L| < \varepsilon. \end{aligned}$$

Prove that $\lim_{x\to 1}(x+2) = 3$

First note that $a = 1$ and $L = 3$ and let $\varepsilon > 0$ be given. Now we observe that

$$\begin{aligned} |f(x) - L| < \varepsilon &\implies 3 - \varepsilon < x + 2 < 3 + \varepsilon \\ &\implies 1 - \varepsilon < x < 1 + \varepsilon \\ &\implies |x - 1| < \varepsilon. \end{aligned}$$

So if we take $\delta = \varepsilon$, then we get

$$\begin{aligned} |x - 1| < \delta &\implies -\delta < x - 1 < \delta \\ &\implies 1 - \delta < x < 1 + \delta \\ &\implies 3 - \delta < x + 2 < 3 + \delta \\ &\implies |\underbrace{x + 2}_{f(x)} - 3| < \delta \\ &\implies |f(x) - L| < \varepsilon. \end{aligned}$$

Since such a δ works for *every* positive ε, this completes the proof.

One-sided Limits

In a similar way we rigorously define $\lim_{x \to a^{\pm}} f(x)$.

One-sided Limits

In a similar way we rigorously define $\lim_{x \to a^{\pm}} f(x)$. We won't write out the definitions here, but we should make sure we recall that

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Properties of Limits

Using the definition one can prove the following properties.

Properties of Limits

Using the definition one can prove the following properties. Suppose $c \in \mathbb{R}$ and suppose the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and are finite.

Properties of Limits

Using the definition one can prove the following properties. Suppose $c \in \mathbb{R}$ and suppose the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and are finite. Then

- $\lim_{x \to a} c = c$
- $\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- $\lim_{x \to a}(c \cdot f(x)) = c \cdot \lim_{x \to a} f(x)$
- $\lim_{x \to a}(f(x) \cdot g(x)) = (\lim_{x \to a} f(x)) \cdot (\lim_{x \to a} g(x))$
- $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

Limits of Compositions

What is $\lim_{x \to a} f(g(x))$?

Limits of Compositions

What is $\lim_{x \to a} f(g(x))$? Well,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) \ldots$$

Limits of Compositions

What is $\lim_{x \to a} f(g(x))$? Well,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) \ldots$$

but we need to make sure f is "continuous at $\lim_{x \to a} g(x)$".

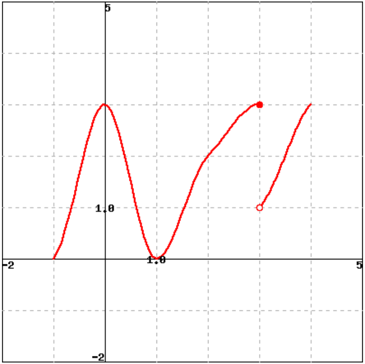
Limits of Compositions

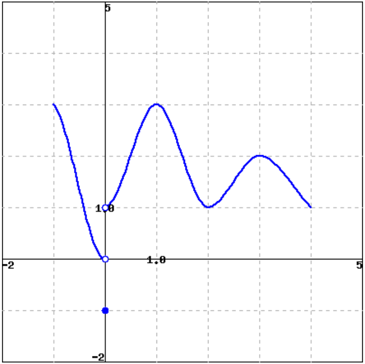
What is $\lim_{x\to a} f(g(x))$? Well,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)\dots$$

but we need to make sure f is "continuous at $\lim_{x\to a} g(x)$". We will talk about continuity soon, but for now we will just explain the intuition in the following examples.

Let f be defined by the red graph and g be defined by blue.





Using the functions on the previous slide (and the limit properties) we can compute the following examples.

Using the functions on the previous slide (and the limit properties) we can compute the following examples.

$$\lim_{x \to 3^+} (f(x)g(x)) =$$

Using the functions on the previous slide (and the limit properties) we can compute the following examples.

$$\lim_{x \to 3^+} (f(x)g(x)) = \left(\lim_{x \to 3^+} f(x) \right) \cdot \left(\lim_{x \to 3^+} g(x) \right) = 1 \cdot 2 = 2$$

Using the functions on the previous slide (and the limit properties) we can compute the following examples.

$$\lim_{x\to 3^+}(f(x)g(x)) = \left(\lim_{x\to 3^+} f(x)\right)\cdot\left(\lim_{x\to 3^+} g(x)\right) = 1\cdot 2 = 2$$

$$\lim_{x\to 0^-}(f(x)+g(x)) =$$

Using the functions on the previous slide (and the limit properties) we can compute the following examples.

$$\lim_{x\to 3^+}(f(x)g(x)) = \left(\lim_{x\to 3^+} f(x)\right)\cdot\left(\lim_{x\to 3^+} g(x)\right) = 1\cdot 2 = 2$$

$$\lim_{x\to 0^-}(f(x)+g(x)) = 3+0 = 3$$

Using the functions on the previous slide (and the limit properties) we can compute the following examples.

$$\lim_{x\to 3^+}(f(x)g(x)) = \left(\lim_{x\to 3^+} f(x)\right)\cdot\left(\lim_{x\to 3^+} g(x)\right) = 1\cdot 2 = 2$$

$$\lim_{x\to 0^-}(f(x)+g(x)) = 3+0 = 3$$

$$\lim_{x\to 0^-} f(g(x)) =$$

Using the functions on the previous slide (and the limit properties) we can compute the following examples.

$$\lim_{x \to 3^+} (f(x)g(x)) = \left(\lim_{x \to 3^+} f(x) \right) \cdot \left(\lim_{x \to 3^+} g(x) \right) = 1 \cdot 2 = 2$$

$$\lim_{x \to 0^-} (f(x) + g(x)) = 3 + 0 = 3$$

$$\lim_{x \to 0^-} f(g(x)) = f\left(\lim_{x \to 0^-} g(x) \right) = f(0) = 3$$

More Examples

$\lim_{x \to -4} \frac{x^2+10x+24}{x+4} =$

More Examples

$\lim_{x \to -4} \frac{x^2+10x+24}{x+4} = 2$

More Examples

$\lim_{x \to -4} \frac{x^2+10x+24}{x+4} = 2$ corrected typo here

More Examples

$\lim_{x \to -4} \frac{x^2+10x+24}{x+4} = 2$ corrected typo here

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) =$

More Examples

$\lim_{x \to -4} \frac{x^2+10x+24}{x+4} = 2$ corrected typo here

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) =$ Does Not Exist

More Examples

$\lim_{x \to -4} \frac{x^2+10x+24}{x+4} = 2$ corrected typo here

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) =$ Does Not Exist

$\lim_{x \to 0} \log x =$

More Examples

$\lim_{x \to -4} \frac{x^2+10x+24}{x+4} = 2$ corrected typo here

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) =$ Does Not Exist

$\lim_{x \to 0} \log x =$ Does Not Exist

More Examples

$\lim_{x \to -4} \frac{x^2+10x+24}{x+4} = 2$ corrected typo here

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) =$ Does Not Exist

$\lim_{x \to 0} \log x =$ Does Not Exist

$\lim_{x \to 0^+} \log x =$

More Examples

$\lim_{x \to -4} \frac{x^2+10x+24}{x+4} = 2$ corrected typo here

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) =$ Does Not Exist

$\lim_{x \to 0} \log x =$ Does Not Exist

$\lim_{x \to 0^+} \log x = -\infty$

More Examples

$\lim_{x \to -4} \frac{x^2+10x+24}{x+4} = 2$ corrected typo here 

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) =$ Does Not Exist

$\lim_{x \to 0} \log x =$ Does Not Exist

$\lim_{x \to 0^+} \log x = -\infty$

$\lim_{x \to -\pi/2} \frac{\sin(\cos x)}{\cos x} =$

More Examples

$\lim_{x \to -4} \frac{x^2+10x+24}{x+4} = 2$ corrected typo here

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) =$ Does Not Exist

$\lim_{x \to 0} \log x =$ Does Not Exist

$\lim_{x \to 0^+} \log x = -\infty$

$\lim_{x \to -\pi/2} \frac{\sin(\cos x)}{\cos x} = 1$

More Examples

$$\lim_{x \to 2} \frac{x-2}{x^2-2x} =$$

More Examples

$$\lim_{x \to 2} \frac{x-2}{x^2-2x} = \frac{1}{2}$$

More Examples

$$\lim_{x \to 2} \frac{x-2}{x^2-2x} = \frac{1}{2}$$

$$\lim_{x \to 4} \frac{x^2-16}{5x^2-17x-12} =$$

More Examples

$$\lim_{x \to 2} \frac{x-2}{x^2-2x} = \frac{1}{2}$$
$$\lim_{x \to 4} \frac{x^2-16}{5x^2-17x-12} = \frac{8}{23}$$

More Examples

$$\lim_{x \to 2} \frac{x-2}{x^2-2x} = \frac{1}{2}$$

$$\lim_{x \to 4} \frac{x^2-16}{5x^2-17x-12} = \frac{8}{23}$$

$$\lim_{h \to 0} \frac{(3+h)^2-9}{h} =$$

More Examples

$$\lim_{x\to 2} \frac{x-2}{x^2-2x} = \frac{1}{2}$$

$$\lim_{x\to 4} \frac{x^2-16}{5x^2-17x-12} = \frac{8}{23}$$

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = 6$$

More Examples

$$\lim_{x \to 2} \frac{x-2}{x^2-2x} = \frac{1}{2}$$

$$\lim_{x \to 4} \frac{x^2-16}{5x^2-17x-12} = \frac{8}{23}$$

$$\lim_{h \to 0} \frac{(3+h)^2-9}{h} = 6$$

We can manipulate the functions in an algebraic way to make limit computations more apparent.

As Time Permits

Consider the function $f(x) = \log(x^2 - x - 2)\ldots$

That's it for today!

That's it for today!