Math 1 Lecture 11

Dartmouth College

Wednesday 10-05-16

Contents

Reminders/Announcements

- WebWork due Friday
- Thursday x-hour trig review

Recall the definition

A sequence is said to **converge** to a **limit** L if for every $\varepsilon > 0$ there exists a positive integer N such that

$$|a_n - L| < \varepsilon$$

for every $n \geq N$.

Recall the definition

A sequence is said to **converge** to a **limit** L if for every $\varepsilon > 0$ there exists a positive integer N such that

$$|a_n - L| < \varepsilon$$

for every $n \geq N$. It is common to package this all together with the following notation:

$$\boxed{\lim_{n\to\infty} a_n = L.}$$

Recall the definition

A sequence is said to **converge** to a **limit** L if for every $\varepsilon > 0$ there exists a positive integer N such that

$$|a_n - L| < \varepsilon$$

for every $n \geq N$. It is common to package this all together with the following notation:

$$\boxed{\lim_{n\to\infty} a_n = L.}$$

When a sequence does not converge we say it **diverges** or "fails to have a limit".

Consider the sequence

$$\{1/(n^3)\}_{n=1}^{\infty}$$

Consider the sequence

$$\{1/(n^3)\}_{n=1}^{\infty}$$

Monotone? Yes. Decreasing.
Bounded? Yes. $M = 1$.
Convergent? Yes. $L = 0$.

Consider the sequence

$$\{1/(n^{-3})\}_{n=1}^{\infty}$$

Consider the sequence

$$\{1/(n^{-3})\}_{n=1}^{\infty}$$

Monotone? Yes. Increasing.
Bounded? No.
Convergent? No.

Consider the sequence

$$\{1/(n^p)\}_{n=1}^{\infty} \quad p \text{ in } \mathbb{R}$$

Consider the sequence

$$\{1/(n^p)\}_{n=1}^{\infty} \quad p \text{ in } \mathbb{R}$$

Monotone? Always monotone. If $p > 0$, then decreasing. If $p < 0$, then increasing. What about when $p = 0$?
Bounded? Only when $p \geq 0$. What's a bound?
Convergent? Only when $p \geq 0$. When $p = 0$, $\lim_{n\to\infty} a_n = 1$.
What about when $p > 0$?

Consider the sequence

$$\{0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots\}$$

Consider the sequence

$$\{0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \dots\}$$

Monotone? Nah.
Bounded? Yep.
Convergent? No way!

Consider the sequence

$$\{0, 1, 0, 0, 1/2, 0, 0, 0, 1/3, 0, 0, 0, 0, 1/4, \dots\}$$

Consider the sequence

$$\{0, 1, 0, 0, 1/2, 0, 0, 0, 1/3, 0, 0, 0, 0, 1/4, \dots\}$$

Monotone? Nope.
Bounded? Yes!
Convergent? Yes. $\lim_{n\to\infty} a_n = 0$.

Consider the sequence

$$\{0.9, 0.99, 0.999, \dots\}$$

Consider the sequence

$$\{0.9, 0.99, 0.999, \dots\}$$

Monotone? Yes.
Bounded? Yes.
Convergent? Yes. Haven't you heard someone say that $0.\overline{9} = 1$?

Consider the sequence

$$\{(-1)^n\}_{n=1}^{\infty}$$

Consider the sequence

$$\{(-1)^n\}_{n=1}^{\infty}$$

Monotone? No.
Bounded? Yes.
Convergent? Nope.

Consider the sequence

$$\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$$

Consider the sequence

$$\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$$

Monotone? No.
Bounded? Yep.
Convergent? Yup.

Consider the sequence

$$\left\{ \frac{\cos(n\pi)}{n} \right\}_{n=1}^{\infty}$$

Consider the sequence

$$\left\{ \frac{\cos(n\pi)}{n} \right\}_{n=1}^{\infty}$$

Monotone? Nope.
Bounded? Yup.
Convergent? Yup.

Consider the sequence

$$\left\{ \frac{n+1}{n-1} \right\}_{n=2}^{\infty}$$

Consider the sequence

$$\left\{ \frac{n+1}{n-1} \right\}_{n=2}^{\infty}$$

Monotone? Yes. Decreasing.
Bounded? Yes.
Convergent? Yes. $L = 1$.

Consider the sequence

$$\left\{ \frac{2n^2 + n + 5}{31n^2 + 100n + 82364} \right\}_{n=1}^{\infty}$$

Consider the sequence

$$\left\{ \frac{2n^2 + n + 5}{31n^2 + 100n + 82364} \right\}_{n=1}^{\infty}$$

Monotone? Yes. Increasing.
Bounded? Yes... by the limit.
Convergent? Yes. $\lim_{n\to\infty} a_n = 2/31$.

Consider the sequence

$$\left\{ \frac{(2n-1)(1-5n)}{2n(n+1)} \right\}_{n=1}^{\infty}$$

Consider the sequence

$$\left\{ \frac{(2n-1)(1-5n)}{2n(n+1)} \right\}_{n=1}^{\infty}$$

Monotone? Yes. Decreasing.
Bounded? Yes. $M = 5$ works. $M = 1000$ also works...
Convergent? Yes. The limit is $2 \cdot (-5)/2 = -5$.

Consider the sequence

$$\left\{ e^{(-n^2)} \right\}_{n=1}^{\infty}$$

Consider the sequence

$$\left\{ e^{(-n^2)} \right\}_{n=1}^{\infty}$$

Monotone? Yes. It is decreasing.
Bounded? Yep. By the first term.
Convergent? The limit is zero.

Consider the sequence

$$\left\{\log_e\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$$

Consider the sequence

$$\left\{\log_e\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$$

Monotone? Yes. Decreasing.
Bounded? Nope.
Convergent? Nope. It diverges to $-\infty$.

Is a convergent sequence necessarily bounded?

Is a convergent sequence necessarily bounded? Yes!

Is a convergent sequence necessarily bounded? Yes!
Is a bounded sequence necessarily convergent?

Is a convergent sequence necessarily bounded? Yes!
Is a bounded sequence necessarily convergent? No!

Is a convergent sequence necessarily bounded? Yes!
Is a bounded sequence necessarily convergent? No! Example?

Is a convergent sequence necessarily bounded? Yes!
Is a bounded sequence necessarily convergent? No! Example?
Is a bounded and monotone sequence necessarily convergent?

Is a convergent sequence necessarily bounded? Yes!
Is a bounded sequence necessarily convergent? No! Example?
Is a bounded and monotone sequence necessarily convergent? Yes!

Is a convergent sequence necessarily bounded? Yes!
Is a bounded sequence necessarily convergent? No! Example?
Is a bounded and monotone sequence necessarily convergent? Yes!
Is a convergent sequence necessarily bounded and monotone?

Is a convergent sequence necessarily bounded? Yes!
Is a bounded sequence necessarily convergent? No! Example?
Is a bounded and monotone sequence necessarily convergent? Yes!
Is a convergent sequence necessarily bounded and monotone? No!

Is a convergent sequence necessarily bounded? Yes!
Is a bounded sequence necessarily convergent? No! Example?
Is a bounded and monotone sequence necessarily convergent? Yes!
Is a convergent sequence necessarily bounded and monotone? No! Example?

Geometric Sequences

A sequence is **geometric** if it is of the form

$$\{a \cdot r^n\}_{n=0}^{\infty}$$

with a, r in $\mathbb{R}$. We call r the **common ratio**.

Consider the sequence

$$\{2 \cdot 3^n\}_{n=0}^{\infty}$$

Consider the sequence

$$\{2 \cdot 3^n\}_{n=0}^{\infty}$$

Monotone? Increasing.
Bounded? No.
Convergent? No.

Consider the sequence

$$\left\{2 \cdot \left(\frac{1}{3}\right)^n\right\}_{n=0}^{\infty}$$

Consider the sequence

$$\left\{2 \cdot \left(\frac{1}{3}\right)^n\right\}_{n=0}^{\infty}$$

Monotone? Yes. Decreasing.
Bounded? Yes. By the first term.
Convergent? Yep. It converges to zero.

For what values of r does a geometric sequence converge?

For what values of r does a geometric sequence converge?
A geometric sequence converges if and only if r in $(-1, 1)$.

Back to the definition

Consider the sequence

$$\{0.9, 0.99, 0.999, 0.9999, \dots\}$$

Back to the definition

Consider the sequence

$$\{0.9, 0.99, 0.999, 0.9999, \dots\}$$

It is *clear* that this sequence converges to 1.

Back to the definition

Consider the sequence

$$\{0.9, 0.99, 0.999, 0.9999, \dots\}$$

It is *clear* that this sequence converges to 1. But how do we *prove* it?

Back to the definition

Consider the sequence

$$\{0.9, 0.99, 0.999, 0.9999, \ldots\}$$

It is *clear* that this sequence converges to 1. But how do we *prove* it? Suppose $\varepsilon = 0.1$.

Back to the definition

Consider the sequence

$$\{0.9, 0.99, 0.999, 0.9999, \dots\}$$

It is *clear* that this sequence converges to 1. But how do we *prove* it? Suppose $\varepsilon = 0.1$. Can we find N such that $|a_n - L| < \varepsilon$ for all $n \geq N$?

Back to the definition

Consider the sequence

$$\{0.9, 0.99, 0.999, 0.9999, \dots\}$$

It is *clear* that this sequence converges to 1. But how do we *prove* it? Suppose $\varepsilon = 0.1$. Can we find N such that $|a_n - L| < \varepsilon$ for all $n \geq N$? Sure! Take $N = 2$.

Back to the definition

Consider the sequence

$$\{0.9, 0.99, 0.999, 0.9999, \dots\}$$

It is *clear* that this sequence converges to 1. But how do we *prove* it? Suppose $\varepsilon = 0.1$. Can we find N such that $|a_n - L| < \varepsilon$ for all $n \geq N$? Sure! Take $N = 2$. What if $\varepsilon = 0.01$?

Back to the definition

Consider the sequence

$$\{0.9, 0.99, 0.999, 0.9999, \dots\}$$

It is *clear* that this sequence converges to 1. But how do we *prove* it? Suppose $\varepsilon = 0.1$. Can we find N such that $|a_n - L| < \varepsilon$ for all $n \geq N$? Sure! Take $N = 2$. What if $\varepsilon = 0.01$? Can we find N then?

Back to the definition

Consider the sequence

$$\{0.9, 0.99, 0.999, 0.9999, \dots\}$$

It is *clear* that this sequence converges to 1. But how do we *prove* it? Suppose $\varepsilon = 0.1$. Can we find N such that $|a_n - L| < \varepsilon$ for all $n \geq N$? Sure! Take $N = 2$. What if $\varepsilon = 0.01$? Can we find N then? Yes!

Back to the definition

Consider the sequence

$$\{0.9, 0.99, 0.999, 0.9999, \dots\}$$

It is *clear* that this sequence converges to 1. But how do we *prove* it? Suppose $\varepsilon = 0.1$. Can we find N such that $|a_n - L| < \varepsilon$ for all $n \geq N$? Sure! Take $N = 2$. What if $\varepsilon = 0.01$? Can we find N then? Yes! So what?

Back to the definition

Consider the sequence

$$\{0.9, 0.99, 0.999, 0.9999, \dots\}$$

It is *clear* that this sequence converges to 1. But how do we *prove* it? Suppose $\varepsilon = 0.1$. Can we find N such that $|a_n - L| < \varepsilon$ for all $n \geq N$? Sure! Take $N = 2$. What if $\varepsilon = 0.01$? Can we find N then? Yes! So what?

A sequence is convergent if we can *always* find an N no matter how small we choose $\varepsilon > 0$.

Consider the sequence

$$\left\{ \tan\left(\frac{\pi}{2} - \frac{1}{n}\right) \right\}_{n=1}^{\infty}$$

Consider the sequence

$$\left\{\tan\left(\frac{\pi}{2} - \frac{1}{n}\right)\right\}_{n=1}^{\infty}$$

Monotone? Yes. Increasing.
Bounded? No.
Convergent? No way!

Consider the sequence

$$\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{\infty}$$

Consider the sequence

$$\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{\infty}$$

Monotone? Decreasing.
Bounded? By the first term...
Convergent? Yes. Notice that

$$\begin{aligned} a_n &= (\sqrt{n+1}-\sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}} \end{aligned}$$

which converges to zero.

OK that's it! Come back during x-hour tomorrow for some trig review/practice if you feel like you need it!