Math 1 Lecture 10

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Monday 10-03-16

Contents

Reminders/Announcements

- WebWork due Wednesday
- Written HW due Wednesday
- Quiz Today

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- WebWork due Wednesday
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- Quiz Today
- But let's talk a bit more about inverse functions first...

$f(x) = \arcsin(x)$

To make $\sin x$ one-to-one we restrict to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then define the **arcsine** function by the following.

$$\arcsin x = y \iff \sin y = x$$

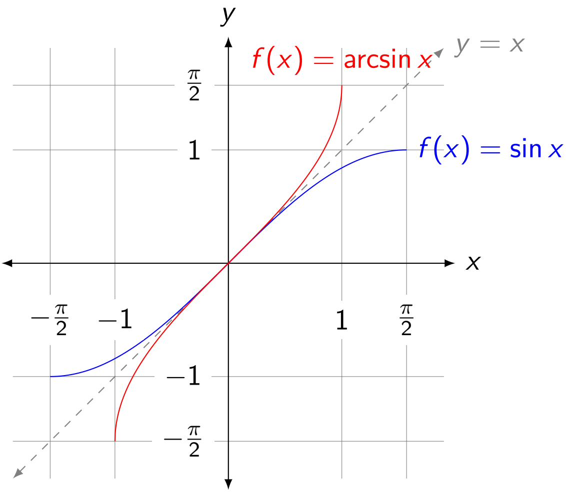
- The domain of $\arcsin x$ is $[-1, 1]$.
- The range of $\arcsin x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- For all x in $[-1, 1]$ we have that

$$\sin(\arcsin x) = x$$

- For all x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ we have that

$$\arcsin(\sin x) = x$$

$f(x) = \arcsin x$



y
$y = x$
$f(x) = \arcsin x$
$\frac{\pi}{2}$
1
$f(x) = \sin x$
x
$-\frac{\pi}{2}$
-1
1
$\frac{\pi}{2}$
-1
$-\frac{\pi}{2}$

$f(x) = \arccos(x)$

To make $\cos x$ one-to-one we restrict to the domain $[0, \pi]$. Then define the **arccosine** function by the following.

$$\arccos x = y \iff \cos y = x$$

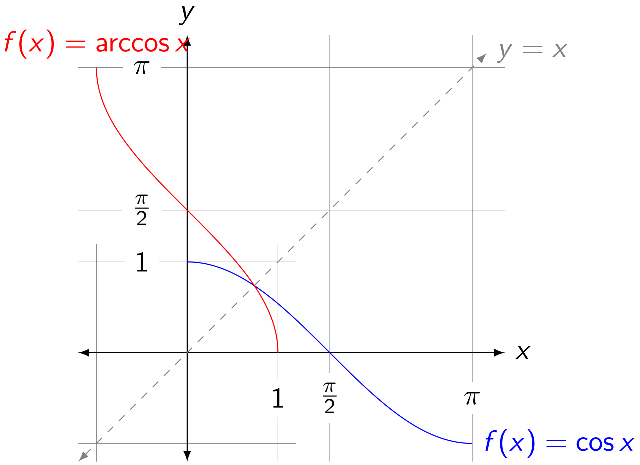
- The domain of $\arccos x$ is $[-1, 1]$.
- The range of $\arccos x$ is $[0, \pi]$.
- For all x in $[-1, 1]$ we have that

$$\cos(\arccos x) = x$$

- For all x in $[0, \pi]$ we have that

$$\arccos(\cos x) = x$$

$f(x) = \arccos x$



$f(x) = \arctan(x)$

To make $\tan x$ one-to-one we restrict to the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$.
Then define the **arctangent** function by the following.

$$\arctan x = y \iff \tan y = x$$

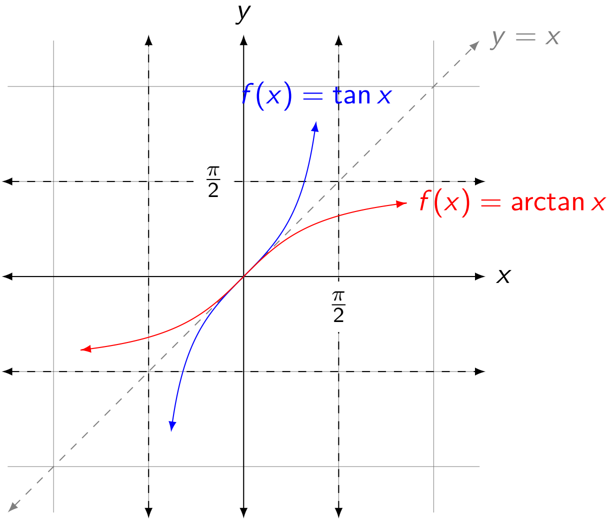
- The domain of $\arctan x$ is $\mathbb{R}$.
- The range of $\arctan x$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- For all x in $\mathbb{R}$ we have that

$$\tan(\arctan x) = x$$

- For all x in $(-\frac{\pi}{2}, \frac{\pi}{2})$ we have that

$$\arctan(\tan x) = x$$

$f(x) = \arctan x$



Please find $\arcsin(\sqrt{2}/2)$.

Please find arcsin($\sqrt{2}/2$).

Solution:

We want to find an angle θ with sin(θ) $= \sqrt{2}/2$ and $-\pi/2 \leq \theta \leq \pi/2$. Why?

Please find arcsin($\sqrt{2}/2$).

Solution:

We want to find an angle θ with sin(θ) $= \sqrt{2}/2$ and $-\pi/2 \leq \theta \leq \pi/2$. Why? Because that is the range of arcsin(x).

Please find $\arcsin(\sqrt{2}/2)$.

Solution:

We want to find an angle θ with $\sin(\theta) = \sqrt{2}/2$ and $-\pi/2 \leq \theta \leq \pi/2$. Why? Because that is the range of $\arcsin(x)$. OK fine, $\theta = \pi/4$.

Please find $\arctan(\sqrt{3})$.

Please find arctan($\sqrt{3}$).

Solution:

We want to find an angle θ with $\tan(\theta) = \sqrt{3}$ and $-\pi/2 \leq \theta \leq \pi/2 \ldots$

Please find $\arctan(\sqrt{3})$.

Solution:

We want to find an angle θ with $\tan(\theta) = \sqrt{3}$ and $-\pi/2 \leq \theta \leq \pi/2 \ldots \theta = \pi/3$.

Please find $\arcsin(\sin(3\pi/4))$.

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Solution:

These functions are inverses of each other, so the answer is obviously $3\pi/4$ right?

Please find arcsin(sin($3\pi/4$)).

Solution:

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Please find $\arcsin(\sin(3\pi/4))$.

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Please find $\arcsin(\sin(3\pi/4))$.

Solution:

These functions are inverses of each other, so the answer is obviously $3\pi/4$ right? No! No! No! No! No! No! No! Why? Because $3\pi/4$ is not in the range of $\arcsin(x)$!

Please find $\arcsin(\sin(3\pi/4))$.

Solution:

These functions are inverses of each other, so the answer is obviously $3\pi/4$ right? No! No! No! No! No! No! No! Why? Because $3\pi/4$ is not in the range of $\arcsin(x)$! With this in mind we see that

$$\arcsin(\sin(3\pi/4)) = \arcsin(\sqrt{2}/2) = \pi/4.$$

Please find $\arcsin(\sin(3\pi/4))$.

Solution:

These functions are inverses of each other, so the answer is obviously $3\pi/4$ right? No! No! No! No! No! No! No! Why? Because $3\pi/4$ is not in the range of $\arcsin(x)$! With this in mind we see that

$$\arcsin(\sin(3\pi/4)) = \arcsin(\sqrt{2}/2) = \pi/4.$$

UGH!

Practice Using the Trigonometry Review Sheet!

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https://math.dartmouth.edu/~m1f16/MATH1Docs/TrigonometryReview.pdf

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It is nicely organized IMHO.

Please find $\sin(\arctan(\sqrt{3}))$.

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Solution:

$$\sin(\arctan(\sqrt{3})) = \sin(\pi/3) = \sqrt{3}/2.$$

Please find tan(arcsin(12/13)).

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Solution: This is a bit more difficult since we can't find θ directly.

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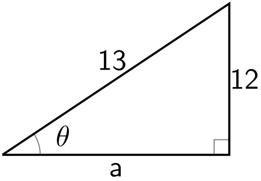
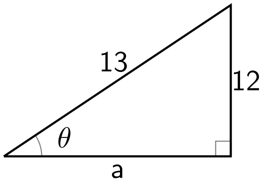


What is *a*?

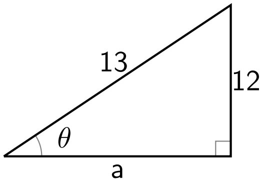
Please find tan(arcsin(12/13)).

Solution: This is a bit more difficult since we can't find θ directly. However, if we let $\theta = \arcsin(12/13)$ we can draw a helpful picture.



What is a? Using Pythagoras we see that $a = 5$.

Please find tan(arcsin(12/13)).
Solution: This is a bit more difficult since we can't find θ directly. However, if we let $\theta = \arcsin(12/13)$ we can draw a helpful picture.



What is a? Using Pythagoras we see that $a = 5$. Now we just have to find $\tan\theta$ for this particular (unknown) θ.

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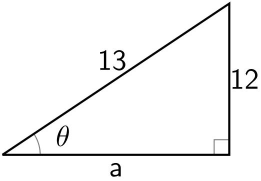

What is a? Using Pythagoras we see that $a = 5$. Now we just have to find $\tan\theta$ for this particular (unknown) θ. We see that $\tan\theta = 12/5$.

Please find a formula for $\cos(\arctan(x))$.

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Solution: Again, we can't find θ directly.

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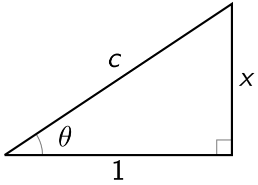
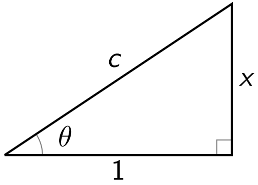


What is c?

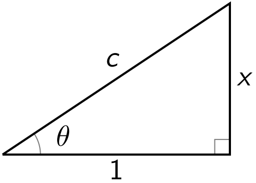
Please find a formula for $\cos(\arctan(x))$.
Solution: Again, we can't find θ directly. However, if we let $\theta = \arctan(x)$ we can draw a helpful picture.



What is c? Using Pythagoras we see that $c = \sqrt{x^2 + 1}$.

Please find a formula for cos(arctan(x)).

Solution: Again, we can't find θ directly. However, if we let $\theta = \text{arctan}(x)$ we can draw a helpful picture.



What is c? Using Pythagoras we see that $c = \sqrt{x^2 + 1}$. Now we just have to find $\cos\theta$ for this particular (unknown) θ.

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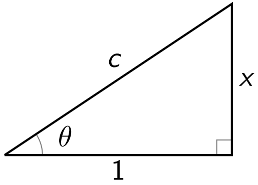

What is c? Using Pythagoras we see that $c = \sqrt{x^2 + 1}$. Now we just have to find $\cos\theta$ for this particular (unknown) θ. We see that $\cos\theta = 1/\sqrt{x^2+1}$.

We will now take 15 minutes for a quiz. . . insert meme here.

Recall. . .

What does it mean for a sequence to be increasing?

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What does it mean for a sequence to be increasing?
$a_{n+1} > a_n$ for every n.

Recall. . .

What does it mean for a sequence to be increasing?
$a_{n+1} > a_n$ for every n. Similarly for decreasing.

Recall...

What does it mean for a sequence to be increasing?
$a_{n+1} > a_n$ for every n. Similarly for decreasing.

What does it mean for a sequence to be *weakly* increasing?

Recall. . .

What does it mean for a sequence to be increasing?
$a_{n+1} > a_n$ for every n. Similarly for decreasing.

What does it mean for a sequence to be *weakly* increasing?
$a_{n+1} \geq a_n$ for every n.

Recall...

What does it mean for a sequence to be increasing?
$a_{n+1} > a_n$ for every n. Similarly for decreasing.

What does it mean for a sequence to be *weakly* increasing?
$a_{n+1} \geq a_n$ for every n. Similarly for weakly decreasing.

Recall...

What does it mean for a sequence to be increasing?
$a_{n+1} > a_n$ for every n. Similarly for decreasing.

What does it mean for a sequence to be *weakly* increasing?
$a_{n+1} \geq a_n$ for every n. Similarly for weakly decreasing.
For example, a constant sequence is weakly increasing but not increasing.

Recall. . .

What does it mean for a sequence to be increasing?
$a_{n+1} > a_n$ for every n. Similarly for decreasing.

What does it mean for a sequence to be *weakly* increasing?
$a_{n+1} \geq a_n$ for every n. Similarly for weakly decreasing.
For example, a constant sequence is weakly increasing but not increasing.

What does it mean for a sequence to be bounded?

Recall...

What does it mean for a sequence to be increasing?
$a_{n+1} > a_n$ for every n. Similarly for decreasing.

What does it mean for a sequence to be *weakly* increasing?
$a_{n+1} \geq a_n$ for every n. Similarly for weakly decreasing.
For example, a constant sequence is weakly increasing but not increasing.

What does it mean for a sequence to be bounded?
There exists $M \geq 0$ such that $|a_n| \leq M$ for every n.

Recall. . .

What does it mean for a sequence to be increasing?
$a_{n+1} > a_n$ for every n. Similarly for decreasing.

What does it mean for a sequence to be *weakly* increasing?
$a_{n+1} \geq a_n$ for every n. Similarly for weakly decreasing.
For example, a constant sequence is weakly increasing but not increasing.

What does it mean for a sequence to be bounded?
There exists $M \geq 0$ such that $|a_n| \leq M$ for every n.

What does it mean for a sequence to be monotone?

Recall...

What does it mean for a sequence to be increasing?
$a_{n+1} > a_n$ for every n. Similarly for decreasing.

What does it mean for a sequence to be *weakly* increasing?
$a_{n+1} \geq a_n$ for every n. Similarly for weakly decreasing.
For example, a constant sequence is weakly increasing but not increasing.

What does it mean for a sequence to be bounded?
There exists $M \geq 0$ such that $|a_n| \leq M$ for every n.

What does it mean for a sequence to be monotone?
Either weakly increasing or weakly decreasing.

Convergence

The definition of convergence can be confusing at first...

Convergence

The definition of convergence can be confusing at first. . .

A sequence is said to **converge** to a **limit** L if for every $\varepsilon > 0$ there exists a positive integer N such that

$$|a_n - L| < \varepsilon$$

for every $n \geq N$.

Convergence

The definition of convergence can be confusing at first...

A sequence is said to **converge** to a **limit** L if for every $\varepsilon > 0$ there exists a positive integer N such that

$$|a_n - L| < \varepsilon$$

for every $n \geq N$.

So before we really try to understand convergence, let's get an intuitive idea about what it means for a sequence to converge or "approach" a limit...

Consider the sequence $\{1/n\}_{n=1}^{\infty}$.

Consider the sequence $\{1/n\}_{n=1}^{\infty}$.
Is this sequence bounded?

Consider the sequence $\{1/n\}_{n=1}^{\infty}$.
Is this sequence bounded? Yes.

Consider the sequence $\{1/n\}_{n=1}^{\infty}$.
Is this sequence bounded? Yes.
Is this sequence decreasing?

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Is this sequence weakly decreasing?

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Is this sequence monotone?

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Consider the sequence $\{1/n\}_{n=1}^{\infty}$.
Is this sequence bounded? Yes.
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Is this sequence weakly decreasing? Yes.
Is this sequence monotone? Yes.
Does this sequence converge?

Consider the sequence $\{1/n\}_{n=1}^{\infty}$.
Is this sequence bounded? Yes.
Is this sequence decreasing? Yes.
Is this sequence weakly decreasing? Yes.
Is this sequence monotone? Yes.
Does this sequence converge? Yes. To $L = 0$.

Consider the sequence $\{1/n\}_{n=1}^{\infty}$.
Is this sequence bounded? Yes.
Is this sequence decreasing? Yes.
Is this sequence weakly decreasing? Yes.
Is this sequence monotone? Yes.
Does this sequence converge? Yes. To $L = 0$. Why?

Consider the sequence $\{1/n\}_{n=1}^{\infty}$.
Is this sequence bounded? Yes.
Is this sequence decreasing? Yes.
Is this sequence weakly decreasing? Yes.
Is this sequence monotone? Yes.
Does this sequence converge? Yes. To $L = 0$. Why? No matter how small an ε we choose, we can always find N so that a_n is within ε of 0 for $n \geq N$.

Consider the sequence $\left\{ \frac{n^2}{n+1} \right\}_{n=1}^{\infty}$.

Consider the sequence $\left\{ \frac{n^2}{n+1} \right\}_{n=1}^{\infty}$.
Is this sequence bounded?

Consider the sequence $\left\{\frac{n^2}{n+1}\right\}_{n=1}^{\infty}$.
Is this sequence bounded? Nope.

Consider the sequence $\left\{ \frac{n^2}{n+1} \right\}_{n=1}^{\infty}$.
Is this sequence bounded? Nope.
Is this sequence decreasing?

Consider the sequence $\left\{\frac{n^2}{n+1}\right\}_{n=1}^{\infty}$.
Is this sequence bounded? Nope.
Is this sequence decreasing? Nope. It's increasing!

Consider the sequence $\left\{\frac{n^2}{n+1}\right\}_{n=1}^{\infty}$.
Is this sequence bounded? Nope.
Is this sequence decreasing? Nope. It's increasing!
Is this sequence weakly decreasing?

Consider the sequence $\left\{\frac{n^2}{n+1}\right\}_{n=1}^{\infty}$.
Is this sequence bounded? Nope.
Is this sequence decreasing? Nope. It's increasing!
Is this sequence weakly decreasing? Nope. It's weakly increasing though!

Consider the sequence $\left\{ \frac{n^2}{n+1} \right\}_{n=1}^{\infty}$.

Is this sequence bounded? Nope.

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Is this sequence monotone?

Consider the sequence $\left\{\frac{n^2}{n+1}\right\}_{n=1}^{\infty}$.
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Consider the sequence $\left\{\frac{n^2}{n+1}\right\}_{n=1}^{\infty}$.
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Consider the sequence $\left\{\frac{n^2}{n+1}\right\}_{n=1}^{\infty}$.

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Consider the sequence $\left\{\frac{n^2}{n+1}\right\}_{n=1}^{\infty}$.
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Is this sequence monotone? Yes.
Does this sequence converge? No. Why?

Consider the sequence $\left\{\frac{n^2}{n+1}\right\}_{n=1}^{\infty}$.
Is this sequence bounded? Nope.
Is this sequence decreasing? Nope. It's increasing!
Is this sequence weakly decreasing? Nope. It's weakly increasing though!
Is this sequence monotone? Yes.
Does this sequence converge? No. Why? The terms get arbitrarily large.

Consider the sequence $\left\{ \frac{5n+6}{9n-1} \right\}_{n=1}^{\infty}$.

Consider the sequence $\left\{\frac{5n+6}{9n-1}\right\}_{n=1}^{\infty}$. Here are some terms of the sequence:

$$
\begin{aligned}
a_1 &= 1.375 \\
a_2 &= 0.941176470588235 \\
a_3 &= 0.807692307692308 \\
a_4 &= 0.742857142857143 \\
&\vdots \\
a_{100} &= 0.562847608453838 \\
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\end{aligned}
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Is this sequence converging?

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Is this sequence converging? Yes!

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Is this sequence converging? Yes! To what?

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\end{aligned}$$

Is this sequence converging? Yes! To what? Well,

$$9/5 = 0.5555555555\ldots$$