Math 1
1st Midterm

September 29, 2016

Name (in block capital letters):

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Instructor (tick one box):

- ☐ **Section 1: M. Musty(10:10am)**
- ☐ **Section 2: E. Sullivan(11:30am)**
- ☐ **Section 3: A. Babei(12:50pm)**
- ☐ **Section 4: M. Dennis(2:10pm)**

Instructions: You are not allowed to provide or receive help of any kind (closed book examination). However, you may ask the instructor for clarification on problems.

1. **Wait** for **signal** to begin.
2. **Write** your **name** in the space provided, and **tick one box** to indicate which section of the course you belong to.
3. Calculators, computers, cell phones, or other computing devices are **not allowed**. In consideration of other students, please **turn off cell phones** or other electronic devices which may be disruptive.
4. Unless otherwise stated, you must **justify your solutions** to receive full credit. Work that is illegible may not be graded. Work that is scratched out will not be graded.

Problem	Score	Possible
1		10
2		20
3		15
4		12
5		12
6		10
7		7
Total		86

1. (**10 points**) Which of the following statements are always true? Write "**T**" for true and "**F**" for false. Your computations will not be graded on this problem.

(a) $\boxed{\text{T}}$ Given the sequence $\left\{\frac{(-1)^n}{n^2+1}\right\}_{n=1}^{\infty}$, the third term is $\frac{-1}{10}$.

(b) $\boxed{\text{F}}$ Recall that a sequence $\{a_n\}$ is bounded if there exists a number M such that $|a_n| \leq M$ for all n. The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is bounded by zero.

(c) $\boxed{\text{F}}$ The function $f(x) = \frac{x^3}{x^2-1}$ is an even function.

(d) $\boxed{\text{T}}$ The function $f(x) = \frac{x^4}{x+1}$ is neither even nor odd.

(e) $\boxed{\text{F}}$ The function $f(x) = (x+3)^2 - 7$ is one-to-one on the interval $[-9, 0]$.

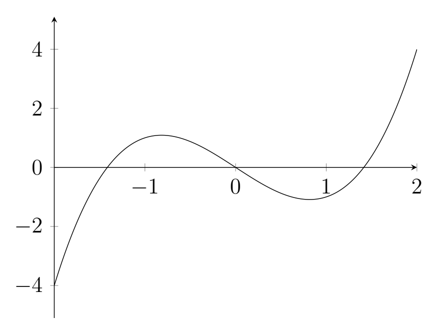
(f) T Every positive decreasing sequence is bounded.

(g) F Every positive increasing sequence is bounded.

(h) T A function never has more than one y-intercept.

(i) F If a function only has one x-intercept, it is one-to-one.

(j) F The following graph has relative minima at $y = -2$ and $y = -4$.



2. (**20 points**) Let $f(x) = \dfrac{1}{\sqrt{x-2}}$, $g(x) = x^2$, $h(x) = x - 2$.

(a) Find the domain and the range of f and g.

Solution:

For f to be defined, we need two things: first $x - 2 \geq 0$ since we can't take square roots of negative numbers, and $\sqrt{x-2} \neq 0$ since we can't divide by 0. Thus $x \geq 2$ and $x \neq 2$, so the domain is $(2, +\infty)$. To find the range, we notice that $f(x)$ only takes positive values since $\sqrt{x-2} > 0$, so the range of f is $(0, +\infty)$. (Alternatively, the graph of f is a shift of the graph of $r(x) = x^{-1/2}$ two units to the right, which has range $(0, +\infty)$).

g is defined for any real number, so the domain is $(-\infty, +\infty)$. Since $x^2 \geq 0$, the range is $[0, +\infty)$

(b) Write the equation of the function $g \circ f$ and find its domain and range.

Solution:

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{\sqrt{x-2}}\right) = \left(\frac{1}{\sqrt{x-2}}\right)^2 = \frac{1}{x-2}$$

Since in this function composition we start first in the domain of f, the domain of $g \circ f$ is contained in $(2, +\infty)$. Since the whole range of f is contained in the domain of g, the domain of $g \circ f$ is the whole domain of f, so it is $(2, +\infty)$.

To find the range, we notice that $\frac{1}{x-2}$ can never be 0. Furthermore, we start in the domain $x > 2$, so $\frac{1}{x-2}$ never reaches negative values either. Therefore, the range is $(0, +\infty)$.

(c) Write the equation of hf, and find its domain and range.

Solution:
$(hf)(x) = \dfrac{x-2}{\sqrt{x-2}} = \sqrt{x-2}$

To find the domain, we keep in mind that we started in both the domain of h, which is $(-\infty, +\infty)$, as well as the domain of f, which is $(2, +\infty)$, so the domain of hf has to be contained in $(2, +\infty)$. But since $\sqrt{x-2}$ is defined on the whole interval $(2, +\infty)$, the domain is $(2, +\infty)$.

The range is $(0, +\infty)$, since we have $x > 2$ in the domain.

(d) Find the equation of f^{-1}, and specify its domain and range.

Solution:
To find the inverse, we first solve $y = \dfrac{1}{\sqrt{x-2}}$ for x in terms of y:

$$\begin{aligned} y &= \frac{1}{\sqrt{x-2}} \\ y^2 &= \frac{1}{x-2} \\ x-2 &= \frac{1}{y^2} \\ x &= \frac{1}{y^2} + 2 \end{aligned}$$

Now, we set $f^{-1}(y) = \dfrac{1}{y^2} + 2$, and we relabel so we get $f^{-1}(x) = \dfrac{1}{x^2} + 2$.

The domain of f^{-1} is the range of f, so it is $(0, +\infty)$.

The range of f^{-1} is the domain of f, which is $(2, +\infty)$.

3. (**15 points**) Samantha would like to create a model for how long it takes her to swim some number of miles. She knows that it takes her 4 hours to swim 3 miles, 6 hours to swim 4 miles, and 8 hours to swim 6 miles. She knows the model should look like a line.

(a) Recall that the Lagrange interpolation formula for two points (x_1, y_1), (x_2, y_2) is

$$f(x) = y_1 \frac{x - x_2}{x_1 - x_2} + y_2 \frac{x - x_1}{x_2 - x_1}.$$

From the statement of the problem, we have three points

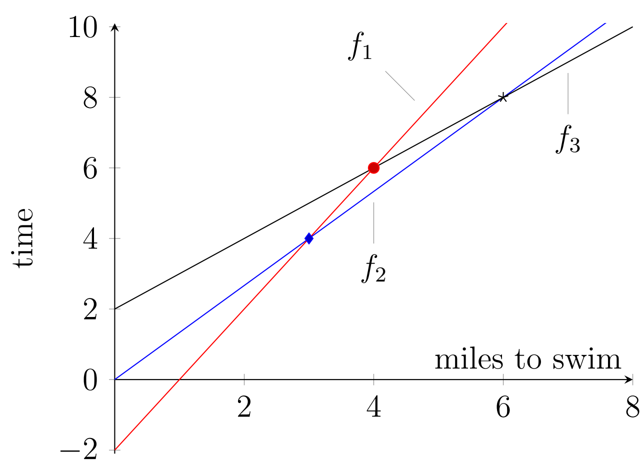
$$p_1 = (3, 4), \qquad p_2 = (4, 6), \qquad p_3(6, 8).$$

Compute the line that goes through p_1 and p_2. Compute the line that goes through p_1 and p_3. Compute the line that goes through p_2 and p_3. Show your work.

Solution:

$$f_1(x) = 4\frac{x-4}{3-4} + 6\frac{x-3}{4-3} = -4(x-4) + 6(x-3) = -4x + 16 + 6x - 18 = 2x - 2$$
$$f_2(x) = 4\frac{x-6}{3-6} + 8\frac{x-3}{6-3} = -\frac{4}{3}(x-6) + \frac{8}{3}(x-3) = -\frac{4}{3}x + 8 + \frac{8}{3}x - 8 = \frac{4}{3}x$$
$$f_3(x) = 6\frac{x-6}{4-6} + 8\frac{x-4}{6-4} = -3(x-6) + 4(x-4) = -3x + 18 + 4x - 16 = x + 2$$

(b) Sketch all three lines and label them.



Solution:

(c) Which line makes the most sense as a model? Explain why.

Solution:

f_2 is the model that makes the most sense. Since the model is measuring how long it takes her to swim. It would make sense that it takes her 0 hours to swim 0 miles, and $f_2(0) = 0$.

4. (**12 points**) Let $f(x)$ be a one-to-one function with domain $[-4, 5]$ and range $[-2, 4]$.

   (a) Write down the domain and range of

   $$3f(-x) + 2.$$

   Show your work. **Solution:** The Domain of f is $[-4, 5]$, so

   $$-4 \leq -x \leq 5 \Rightarrow -5 \leq x \leq 4.$$

   The range of f is $[-2, 4]$, so

   $$-2 \leq f(x) \leq 4 \Rightarrow -6 \leq 3f(x) \leq 12 \Rightarrow -4 \leq 3f(x) + 2 \leq 14.$$

   Thus the domain and range of $3f(-x) + 2$ are $[-5, 4]$ and $[-4, 14]$ respectively.

   (b) Write down the domain and range of

   $$2f^{-1}(x + 1) - 3.$$

   Show your work.

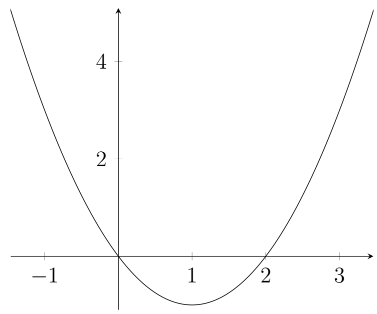
   Solution:
   We are not dealing with f but with f^{-1}. The domain and range of f^{-1} are $[-2, 4]$ and $[-4, 5]$ respectively. Thus

   $$-2 \leq x + 1 \leq 4 \Rightarrow -3 \leq x \leq 3.$$

   And
   $$-4 \leq f^{-1}(x) \leq 5 \Rightarrow -8 \leq 2f^{-1}(x) \leq 10 \Rightarrow -11 \leq 2f^{-1}(x) - 3 \leq 7.$$
   Thus the domain and range of $2f^{-1}(x + 1) - 3$ are $[-3, 3]$ and $[-11, 7]$ respectively.

5. (**12 points**) Consider the following classes of functions: **linear**, **power**, **polynomial**, **rational**. For each of the following graphs, write down which classes (if any) it belongs to, and which ones (if any) it doesn't belong to. Note that all four classes should be written down for each graph.

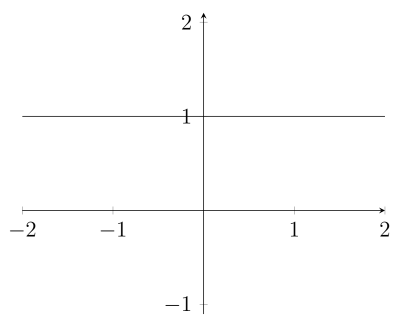
(a)



Belongs to:
Polynomial, **Rational**

Doesn't belong to:
Linear, **Power**

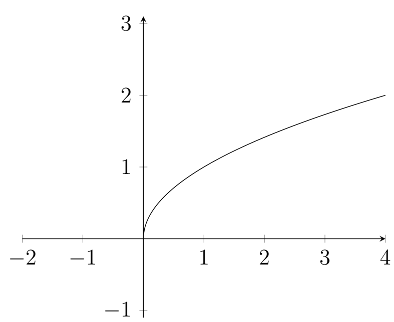
(b)



Belongs to:
Linear , **Power** , **Polynomial**, **Rational**

Doesn't belong to:

(c)



Belongs to:
Power

Doesn't belong to:
Linear, **Polynomial** , **Rational**

6. (**10 points**) Show all your work for the following problems.

(a) Solve for x in the equation
$$16^{(2^x)} = 2^{(4^x)}.$$

Solution: We first take log_2 of both sides to get

$$\begin{aligned} \log_2\left(16^{(2^x)}\right) &= \log_2\left(2^{(4^x)}\right) \\ 2^x \log_2(16) &= 4^x \log_2(2) \\ 4 \cdot 2^x &= 4^x. \end{aligned}$$

Take the log_2 of each side again to get:

$$\begin{aligned} \log_2(4 \cdot 2^x) &= \log_2(4^x) \\ \log_2(4) + \log_2(2^x) &= x \log_2(4) \\ 2 + x\log_2(2) &= 2x \\ 2 + x &= 2x \\ x &= 2 \end{aligned}$$

(b) Solve for x in the equation

$$\log_3(x) + \log_3(x+1) = \log_3(2).$$

Solution: Combine the logs on the left hand side to get

$$\log_3(x(x+1)) = \log_3(2).$$

Raise 3 to the powers of both sides to get

$$x(x+1) = 2.$$

The quadratic becomes
$$x^2 + x - 2 = (x-1)(x+2)$$

And so $x = 1, -2$. But non-positive numbers are not in the domain of logarithmic functions. So we can have $\log_3(-2)$. So -2 is not a solution. Thus $x = 1$ is the only solution.

7. (**7 points**) Let $f(x) = 3x^2 - 5x + 2$.

(a) Find the average rate of change of f on the interval $[0, 1]$.

Solution:
We use the average rate of change formula $\dfrac{f(b) - f(a)}{b - a}$, which is the same as the formula for finding the slope of the secant line going through $(a, f(a)), (b, f(b))$. In this case, we want the secant line going through $(0, f(0))$ and $(1, f(1))$:

$$\frac{f(1) - f(0)}{1 - 0} = \frac{(3 \cdot 1 - 5 \cdot 1 + 2) - (3 \cdot 0 - 5 \cdot 0 + 2)}{1 - 0} = \frac{0 - 2}{1 - 0} = -2$$

(b) Find the average rate of change of f on the interval $[1, 2]$.

Solution: Now we want the slope of the secant line going through the points $(1, f(1)), (2, f(2))$, so we have

$$\frac{f(2) - f(1)}{2 - 1} = \frac{(3 \cdot 2^2 - 5 \cdot 2 + 2) - (3 \cdot 1 - 5 \cdot 1 + 2)}{2 - 1} = \frac{4 - 0}{2 - 1} = 4$$