MATH 1 Homework 8 Solutions

Assigned November 2nd, due November 9th

1. Let $xy - 4 = 4y^2$.

   (a) Use implicit differentiation to find $\frac{dy}{dx}$.
   Solution: First take the derivative of both sides: $\frac{dx}{dx}y + \frac{dy}{dx}x = 8y\frac{dy}{dx}$. Now we solve for $\frac{dy}{dx}$: $y = (8y - x)\frac{dy}{dx}$, so
   $$\frac{dy}{dx} = \frac{y}{8y - x}.$$

   (b) For the following points, does there exist a tangent line at that point? If so, find the equation of the tangent line.

      i. $(17, 4)$
      Solution: To see if a tangent line exists at that point, we need to see if that point is on the curve. Plugging in to the left hand side gives us $17 \cdot 4 - 4 = 64$, and plugging in to the right hand side gives us $4(4^2) = 64$. Since the two sides equal each other, that point is on the curve, so we can find a tangent line. To find the slope, we plug the point into our derivative that we found in part (a) to get $\frac{4}{15}$. We can then use point-slope form to find $y - 4 = \frac{4}{15}(x - 17)$, so $y = \frac{4}{15}x - \frac{8}{15}$.

      ii. $(10, 2)$
      Solution: To see if a tangent line exists at that point, we need to see if that point is on the curve. Plugging in to the left hand side gives us $10 \cdot 2 - 4 = 16$, and plugging in to the right hand side gives us $4(2^2) = 16$. Since the two sides equal each other, that point is on the curve, so we can find a tangent line. To find the slope, we plug the point into our derivative that we found in part (a) to get $\frac{2}{6} = \frac{1}{3}$. We can then use point-slope form to find $y - 2 = \frac{1}{3}(x - 10)$, so $y = \frac{1}{3}x - \frac{4}{3}$.

      iii. $(3, 3)$
      Solution: To see if a tangent line exists at that point, we need to see if that point is on the curve. Plugging in to the left hand side gives us $3 \cdot 3 - 4 = 5$, and plugging in to the right hand side gives us $4(3^2) = 36$. Since the two sides do not equal each other, this point is not on the curve, so we cannot find the tangent line at this point.

2. Let $x^4 - 3y^2 = 2xy$. Use implicit differentiation to solve for the following:

   (a) $\frac{dy}{dx}$
   Solution: We begin by taking the derivative of both sides: $4x^3\frac{dx}{dx} - 6y\frac{dy}{dx} = 2(y\frac{dx}{dx} + x\frac{dy}{dx})$. Now we solve for $\frac{dy}{dx}$: $4x^3 - 2y = (2x + 6y)\frac{dy}{dx}$, so
   $$\frac{dy}{dx} = \frac{4x^3 - 2y}{2x + 6y}.$$

(b) $\frac{dx}{dy}$

Solution: We begin by taking the derivative of both sides. This time, however, we are interested in taking the derivative with respect to y, not with respect to x. This means that our default variable should be y, not x. Thus we get $4x^3\frac{dx}{dy} - 6y\frac{dy}{dy} = 2(y\frac{dx}{dy} + x\frac{dy}{dy})$. Now we solve for $\frac{dx}{dy}$: $(4x^3 - 2y)\frac{dx}{dy} = 2x + 6y$, so

$$\frac{dx}{dy} = \frac{2x + 6y}{4x^3 - 2y}.$$

3. Use implicit differentiation to show that $\frac{d}{dx}\text{arcsec}(x) = \frac{1}{x\sqrt{x^2-1}}$. (*Hint:* $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$. *You do not need to show this.*)

Solution: Let $y = \text{arcsec}(x)$, then $\sec(y) = x$ and we are looking for $\frac{dy}{dx}$. If we take the derivative of both sides, we get $\sec(y)\tan(y)\frac{dy}{dx} = 1\frac{dx}{dx}$. Solving for $\frac{dy}{dx} = \frac{1}{\sec(y)\tan(y)}$. Now we need to get this in terms of x. We know that $\sec(y) = x$, so we can replace $\sec(y)$ to get $\frac{dy}{dx} = \frac{1}{x\tan(y)}$. Now we just need to replace $\tan(y)$. We can do this by using the Pythagorean Identity: $\cos^2(y)+\sin^2(y) = 1$. If we divide everything by $\cos^2(y)$, we get $1+\tan^2(y) = \sec^2(y)$. Solving for $\tan(y)$ gives us $\tan(y) = \sqrt{\sec^2(y) - 1} = \sqrt{x^2 - 1}$. Thus we get

$$\frac{d}{dx}\text{arcsec}(x) = \frac{1}{x\sqrt{x^2 - 1}}.$$

4. Let $y = x^x$. Using natural logarithms and implicit differentiation, find $\frac{dy}{dx}$ in terms of x. (*Hint: start by taking the natural logarithm of both sides.*)

Solution: Let $y = x^x$. First, we lake natural logarithm on both sides:

$$\ln(y) = \ln(x^x),$$

so

$$\ln(y) = x\ln(x)$$

Now, we differentiate both sides with respect to x:

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}x\ln(x)$$

By the chain rule on the LHS and the product rule on the RHS, this is equivalent to

$$\frac{d\ln(y)}{dy}\frac{dy}{dx} = \ln(x) + 1$$

$$\frac{1}{y}\frac{dy}{dx} = \ln(x) + 1$$

$$\frac{dy}{dx} = \frac{\ln(x) + 1}{1/y}$$

Since at the beginning we had $y = x^x$, the above reduces to

$$\frac{dy}{dx} = x^x(\ln(x) + 1).$$

5. (a) Explain how we know that $3x^4 - 8x^3 + 2 = 0$ has a root in the interval $[2, 3]$.

Solution: We know this by the Intermediate Value Theorem: let $f(x) = 3x^4 - 8x^3 + 2$, then f is continuous. Since $f(2) = -14$ and $f(3) = 29$, there must be some c in the interval $(2, 3)$ such that $f(c) = 0$.

(b) Starting with $x_0 = 3$, do 3 iterations of Newton's method to approximate the root to three decimal places. Use a calculator, but write down the formulas for each iteration. Show all your work.

Solution: Since $f(x) = 3x^4 - 8x^3 + 2$, we have $f'(x) = 12x^3 - 24x^2$. Starting with $x_0 = 3$, we have

$$x_1 = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{29}{108} \approx 2.731$$

$$x_2 = 2.731 - \frac{f(2.731)}{f'(2.731)} \approx 2.640$$

$$x_3 = 2.640 - \frac{f(2.640)}{f'(2.640)} \approx 2.630$$

Our approximation is that the root is ≈ 2.630.