MATH 1 Homework 6

Assigned October 19th, due October 26th

1. (a) Use the limit definition to find the slope of the tangent line to $f(x) = x - x^2$ at the point $(1, 0)$.

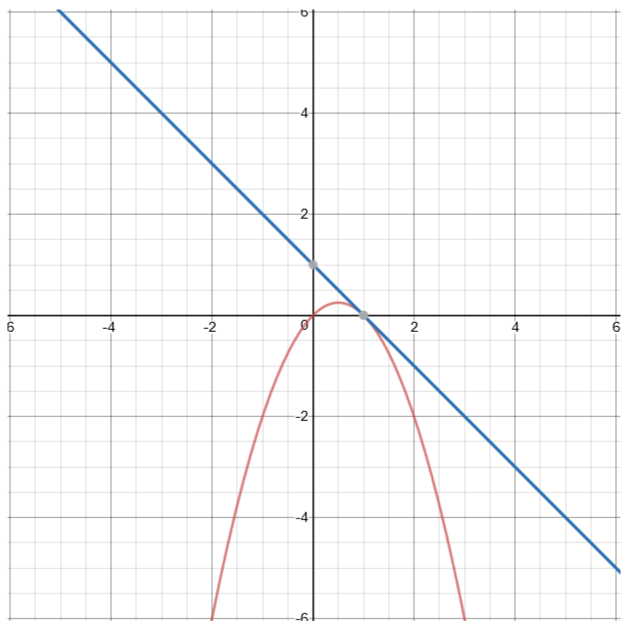
   Solution: We know that $m = \lim_{h\to 0} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x - x^2 - 0}{x - 1} = \lim_{x\to 1} \frac{-x(-1 + x)}{x - 1} = \lim_{x\to 1} -x = -1$.

   (b) Find an equation of the tangent line at the point $(1, 0)$.

   Solution: If we use point-slope form, we get $y - y_1 = m(x - x_1)$, which in this case is $y - 0 = -1(x - 1)$, so we have $y = 1 - x$.

   (c) Graph both $f(x)$ and the tangent line at the point $(1, 0)$. Be sure to label which is which.

   Solution: Here, $f(x)$ is red, and the tangent line is blue.



2. (a) Use the limit definition to find the derivative of $f(x) = x^2 - 4x$.

   Solution: We have

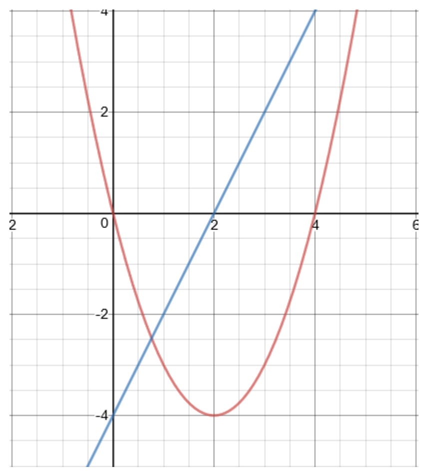
$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{x\to 0} \frac{(x^2 + 2xh + h^2) - 4(x + h) - (x^2 - 4x)}{h} \\
&= \lim_{x\to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x}{h} \\
&= \lim_{x\to 0} \frac{2xh + h^2 - 4h}{h} \\
&= \lim_{x\to 0} \frac{h(2x + h - 4)}{h} \\
&= \lim_{x\to 0} 2x + h - 4 \\
&= 2x - 4
\end{aligned}
$$

(b) Check your work using the power rule.

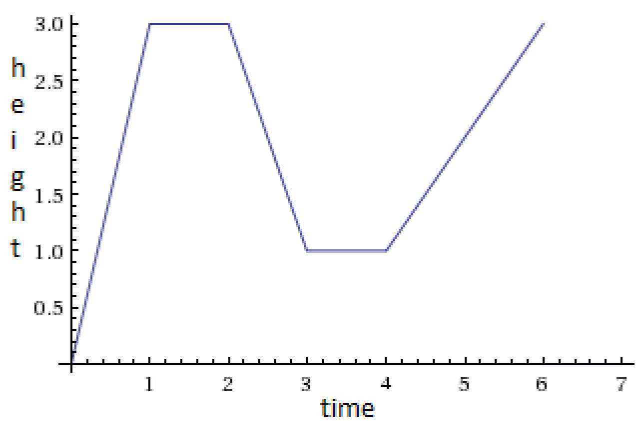
Solution: The power rule says that the derivative of x^n is nx^{n-1}, so the derivative of x^2 is $2x$ and the derivative of $4x$ is 4. Subtracting these gives us the derivative of $f(x)$, so $f'(x) = 2x - 4$.

(c) Graph f and f' (label which is which). Why does it make sense that f' is positive, negative, and zero for the x values that it is (when compared to the graph of f)?

Solution: Below is the graph of f (red) and f' (blue). f' is positive where the graph of f is increasing (and so has positive slope), negative where the graph of f is decreasing (and so has negative slope), and zero where the graph of f is switching from increasing to decreasing (and so has 0 slope).



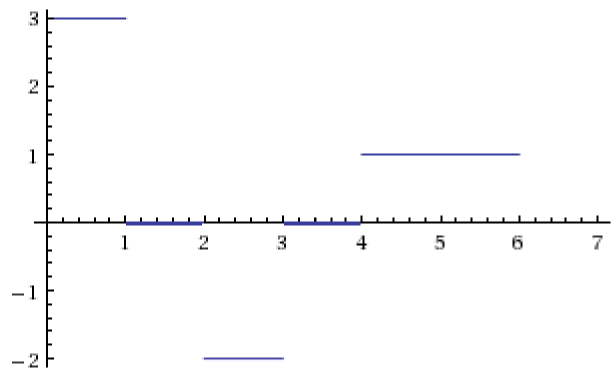
3. (a) A particle starts by moving up along a vertical line; the graph below is of the height of the particle relative to time. When is the particle moving up? Down? Standing still?



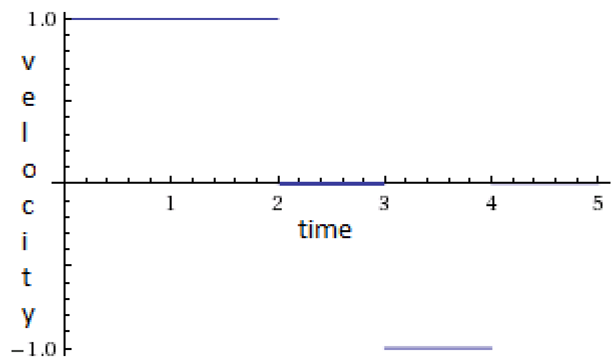
Solution: The particle is moving up when distance along the line is increasing (i.e. when the function is increasing), so when $0 \leq x \leq 1$ and $4 \leq x \leq 6$. It is moving down when the distance moved along the line is decreasing (since it's moving back the way it came), so when $2 \leq x \leq 3$. It is standing still when the distance moved isn't changing, so when $1 \leq x \leq 2$ and $3 \leq x \leq 4$.

(b) Draw a graph of the velocity of the particle relative to time.

Solution:



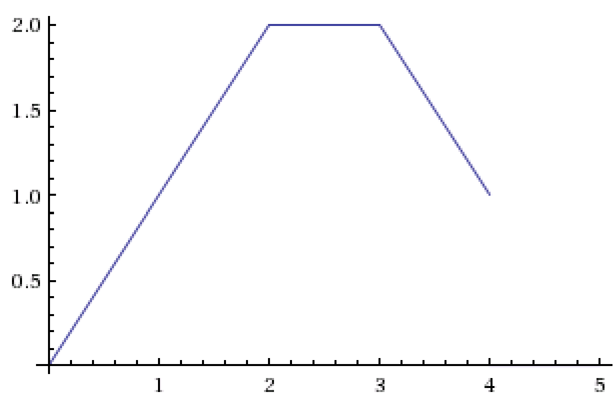
(c) A second particle starts by moving up along a vertical line; the graph of its velocity relative to time is shown below. When is the particle moving up? Down? Standing still?



Solution: It is moving up when the velocity is positive, so when $0 \leq x \leq 2$. It is moving down when the velocity is negative, so when $3 \leq x \leq 4$. It is standing still when velocity is 0, so when $2 \leq x \leq 3$.

(d) Draw a graph of the position function.

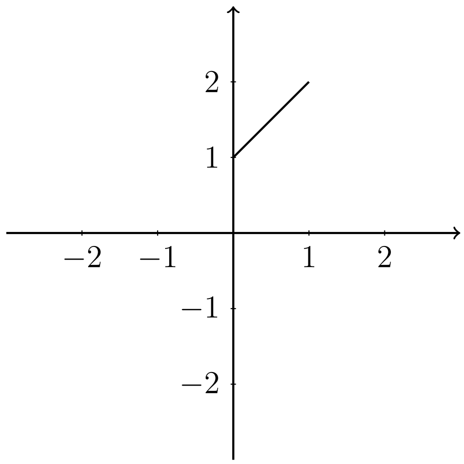
Solution:



4. Draw a graph of a function that is:

(a) positive with positive derivative.

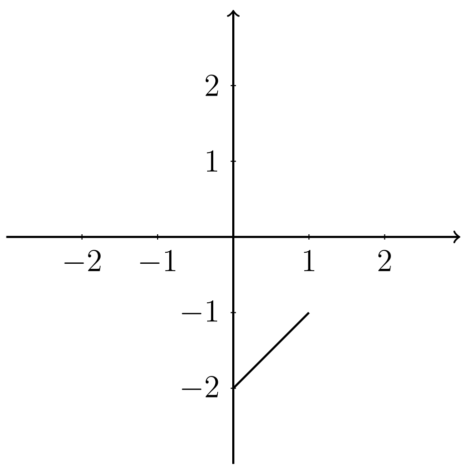
Solution:



In general, we are looking for an increasing function with positive values (we need increasing so the slope of the tangent line at any point in the interval $[0, 1]$ on its graph is positive). Let $f(x) = x + 1$. Then for $0 \le x \le 1$, we have $1 \le f(x) \le 2$, so $f(x)$ is positive on the interval $[0, 1]$, and its derivative is $f'(x) = 1$ which is also positive.

(b) negative with positive derivative.

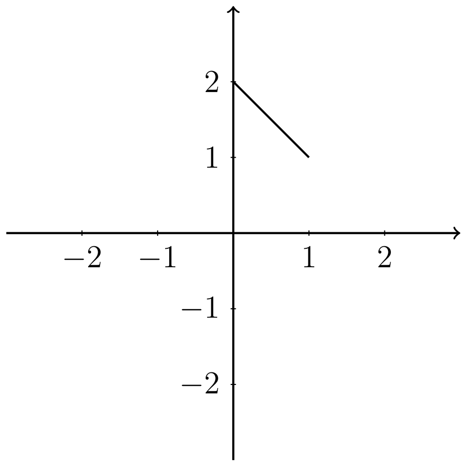
Solution:



We are still looking for an increasing function, but with negative values. Let $g(x) = x - 2$. Then for $0 \le x \le 1$, we have $-2 \le g(x) \le 2$, so $g(x)$ is negative on the interval $[0, 1]$ and its derivative is $g'(x) = 1$ is positive.

(c) positive with negative derivative.

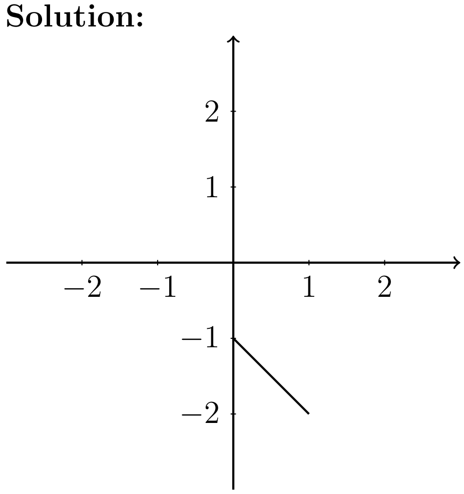
Solution:



Now we are looking for a decreasing function with positive values (decreasing so the slope of the tangent line at any point in the interval $[0, 1]$ on its graph is negative). Let $h(x) = -x + 2$. Then for $0 \leq x \leq 1$, we have $1 \leq h(x) \leq 2$, so $h(x)$ is positive on the interval $[0, 1]$, and its derivative is $h'(x) = -1$ which is negative.

(d) negative with negative derivative.

Solution:



We are still looking for a decreasing function, but with negative values. Let $j(x) = -x-1$. Then for $0 \leq x \leq 1$, we have $-1 \leq g(x) \leq -2$, so $j(x)$ is negative on the interval $[0, 1]$ and its derivative is $j'(x) = -1$ is negative.

5. Compute the following derivatives using the rules for computing derivatives. Show your work.

(a) $g'(3)$ for $g(x) = 3$

Solution: $g(x) = 3$, so $g'(x) = 0$. Therefore $g'(3) = 0$.

(b) $h'(-1)$ for $h(x) = 6e^x + x(x - 1)$

Solution: $h(x) = 6e^x + x(x - 1) = 6e^x + x^2 - x$, so $h'(x) = 6e^x + 2x - 1$. Then $h'(-1) = 6e^{-1} - 2 - 1 = 6e^{-1} - 3$.

(c) $j'(0)$ for $j(x) = 3x^6 + x^2 + x - 5e^x$

Solution: $j(x) = 3x^6+x^2+x-5e^x$, so $j'(x) = 18x^5+2x+1-5e^x$. Thus $j'(0) = 1-5 = -4$.

(d) $k'(-2)$ for $k(x) = x^3 - 3x + 5$

Solution: $k(x) = x^3 - 3x + 5$, so $k'(x) = 3x^2 - 3$. Therefore $k'(-2) = 3 \cdot 4 - 3 = 9$.