MATH 1 Homework 4

Assigned October 5th, due October 12th

1. Do the following sequences converge? If so, what to?

   (a) $\{\frac{1}{n^2}\}_{n=1}^{\infty}$
   Solution: Yes, to zero.

   (b) $\{\cos(n\pi)\}_{n=1}^{\infty}$
   Solution: No, the sequence is equivalent to $\{(-1)^n\}$.

   (c) $\{n^3\}_{n=1}^{\infty}$
   Solution: No, the sequence goes to infinity.

   (d) $\{\frac{2n-2}{n}\}_{n=1}^{\infty}$
   Solution: Yes, the degrees of the top and bottom are the same, so we take the ratio of the leading coefficients. It converges to 2.

   (e) $\{\sin^2(n) + \cos^2(n)\}_{n=1}^{\infty}$
   Solution: By the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$ for any θ. Thus this is the constant sequence 1, so it converges to 1.

2. For the following scenarios, give an example or explain why it can't happen. Can a sequence converge if it is...

   (a) monotone and bounded?

   Solution: Yes, a sequence always converges if it is monotone and bounded. For example $\{\frac{1}{n}\}$.

   (b) monotone and unbounded?

   Solution: No, convergent sequences are always bounded. If it is monotone and unbounded, then the sequence must go to either positive or negative infinity.

   (c) bounded but not monotone?

   Solution: Yes, $\{\frac{(-1)^n}{n}\}$.

3. Let $\{a_n\} = \{0, x^2, 0, x^4, 0, x^6, \dots\}$. If $x = \frac{1}{3}$, the sequence converges. If $x = 2$, the sequence does not converge. Find all values of x for which the sequence converges. Explain why the sequence converges for the values of x that you found, and explain why it does not converge for other values of x.

   Solution: The sequence converges for $(-1, 1)$. If the sequence converges, it must converge to 0 since every other term is 0. If we plug in a fraction, as the powers increase, the fraction will become smaller, so the sequence will converge to zero. If we plug in a positive number larger than 1, the even terms in the sequence will increase to infinity, while the odd terms will stay at zero. If we plug in 1 or -1, the sequence will bounce back and forth between 0 and 1. If we plug in a negative number smaller than -1, the even terms in the sequence will decrease to negative infinity, while the odd terms will stay at zero.

4. Let $\{a_n\}_{n=1}^{\infty} = \{3^n\}_{n=1}^{\infty}$.

(a) Find a sequence $\{b_n\}_{n=1}^{\infty}$ such that the product sequence $\{a_n b_n\}_{n=1}^{\infty}$ converges to 0. Explain why your answer converges to 0.

Solution: In this problem we will use the fact that $\{r^n\}_{n=1}^{\infty}$ converges to 0 if $-1 < r < 1$, it converges to 1 when $r = 1$, and it doesn't converge for any other real value. Depending on the limit that we want, we will pick an appropriate r. Let $\{b_n\}_{n=1}^{\infty} = \{\frac{1}{4^n}\}_{n=1}^{\infty}$. Then $a_n b_n = \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$, so $\{a_n b_n\}_{n=1}^{\infty} = \{\left(\frac{3}{4}\right)^n\}_{n=1}^{\infty}$ converges to 0 as n goes to ∞ since $-1 < \frac{3}{4} < 1$. In fact, any sequence $\{\frac{1}{t^n}\}$ would work as long as $t > 3$ or $t < -3$.

(b) Find a sequence $\{c_n\}_{n=1}^{\infty}$ such that the quotient sequence $\{\frac{a_n}{c_n}\}_{n=1}^{\infty}$ converges to 1. Explain why your answer converges to 1.

Solution: Let $\{c_n\}_{n=1}^{\infty} = \{3^n\}_{n=1}^{\infty}$. Then $\frac{a_n}{c_n} = \frac{3^n}{3^n} = 1$, so $\{\frac{a_n}{c_n}\}_{n=1}^{\infty} = \{1\}_{n=1}^{\infty}$ converges to 1 since all its terms are 1.

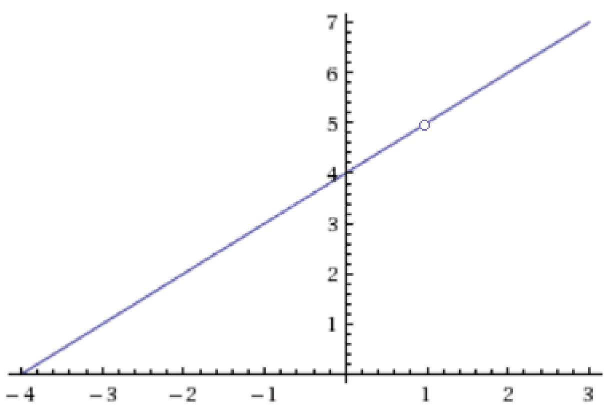
(c) Find a sequence $\{d_n\}_{n=1}^{\infty}$ such that the quotient sequence $\{\frac{a_n}{d_n}\}_{n=1}^{\infty}$ does not converge. Explain why it doesn't converge.

Solution: We can pick $\{d_n\}_{n=1}^{\infty} = \{2^n\}_{n=1}^{\infty}$. Then $\frac{a_n}{d_n} = \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$, so $\{\frac{a_n}{d_n}\}_{n=1}^{\infty} = \{\left(\frac{3}{2}\right)^n\}_{n=1}^{\infty}$ does not converge since $\frac{3}{2} > 1$. In fact, any $\{d_n\}_{n=1}^{\infty} = \{s^n\}_{n=1}^{\infty}$ would work as long as $-3 \leq t < 0$ or $0 < t < 3$.

5. Either plot the graphs of each of the following functions or approximate them numerically in order to guess the following limits. Show your work.

(a) $\displaystyle\lim_{x \to 1} \frac{x^2 + 3x - 4}{x - 1}$

Solution:



By examining the graph of the function above, we see that $\lim_{x \to 1} = 5$. Another approach to get the graph of the function is the following: we can factor the numerator $x^2 + 3x - 4 = (x-1)(x+4)$, so $f(x) = \frac{(x-1)(x+4)}{x-1}$, and by cancelling we get that $f(x) = x + 4$ when $x \neq 1$. Therefore, we just get the line $x + 4$ wth a point at $x = 1$ missing, and the limit is $1 + 4 = 5$.

(b) $\displaystyle\lim_{x \to 1} [\ln(-x^2 + 4x - 3) - \ln(x - 1)]$

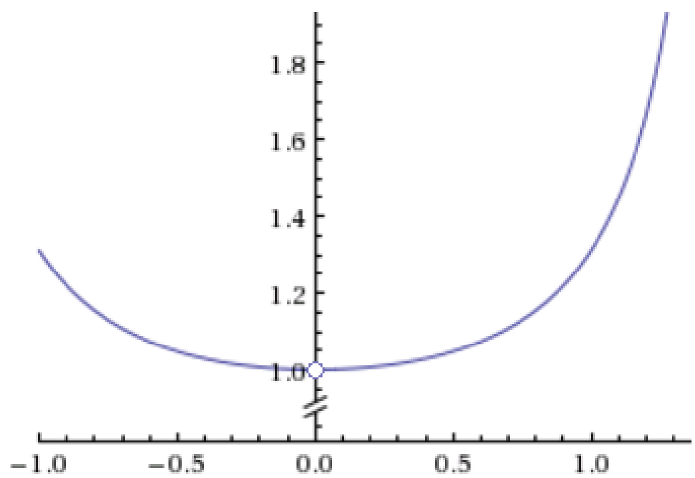
Solution: While we can just graph the function and guess the value, we can use a similar trick from part (a) and factor $-x^2 + 4x - 3 = -(x-1)(x-3)$. Then

$$\ln(-x^2 + 4x - 3) - \ln(x - 1) = \ln \frac{-x^2 + 4x - 3}{x - 1} = \ln \frac{-(x-1)(x-3)}{x - 1}.$$

As we cancel out the factor $x-1$, we get that close to $x=1$, the graph looks like the graph of $\ln(-x+3)$, so $\lim_{x\to 1}\ln(-x+3)=\ln(-1+3)=\ln(2)$.

(c) $\lim_{x\to 0}\dfrac{\sin(x)\tan(x)}{x^2}$

Solution: Graphing the function we get



and we can see that as x gets closer to 0, the value of the function gets closer to 1, so

$$\lim_{x\to 0}\frac{\sin(x)\tan(x)}{x^2}=1.$$

Alternatively, we can plug in values of x that get closer and closer to 0, and see if we observe a trend. For example, $f(0.1)\approx 1.0016$, $f(-0.1)\approx 1.0016$, $f(0.01)\approx 1.00002$, $f(-0.01)\approx 1.00002$ etc. As we see, the values seem to get closer and closer to 1 as x gets closer and closer to 0, so we guess the limit is 1.

6. What are the asymptotes of the following functions? For parts (a) and (b), sketch the function.

(a) $\dfrac{x^2+1}{x^2}$

Solution: We get the horizontal asymptotes by looking at both limits as $x\to-\infty$ and $x\to+\infty$. Thus,

$$\lim_{x\to+\infty}\frac{x^2+1}{x^2}=\lim_{x\to+\infty}\left(\frac{x^2}{x^2}+\frac{1}{x^2}\right)=\lim_{x\to+\infty}\left(1+\frac{1}{x^2}\right)=1,$$

since the term $\frac{1}{x^2}$ gets negligible as x grows very large.

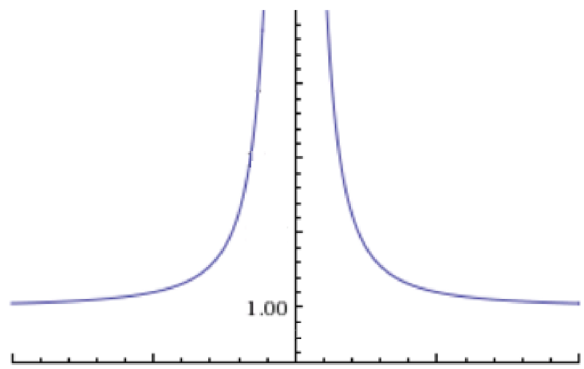
Similarly,

$$\lim_{x\to-\infty}\frac{x^2+1}{x^2}=\lim_{x\to-\infty}\left(\frac{x^2}{x^2}+\frac{1}{x^2}\right)=\lim_{x\to-\infty}\left(1+\frac{1}{x^2}\right)=1,$$

so the only horizontal asymptote is $y=1$.

To find vertical asymptotes, we check the limit near points where the function is not defined. The only such point is $x=0$, so we need to find $\lim_{x\to 0}\dfrac{x^2+1}{x^2}$. To find whether

the values of the function grow large in magnitude and if they do, whether they have positive or negative sign, we plug in some values of x close to 0: at $x = 0.1$, the value of the function is 101; at $x = -0.1$, the value is also 101. At $x = 0.01$ and $x = -0.01$, the value of the function is 10001 . As the values grow large in magnitude and are positive, it is safe to assume that $\lim_{x\to 0} \frac{x^2+1}{x^2} = +\infty$, and we have one vertical asyptote $x = 0$.
Due to the two asymptotes we found above, we can sketch the function, which will look the following way:



(b) $\frac{5x}{3x+3}$

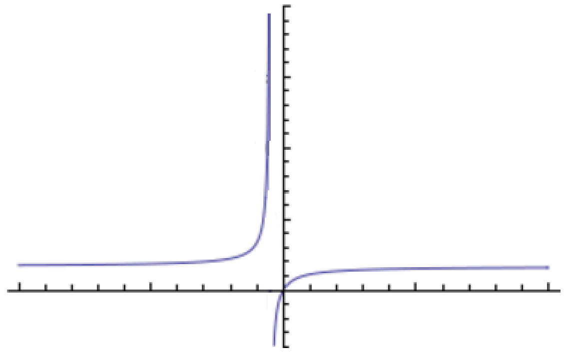
Solution: Horizontal asymptotes:

$$\lim_{x\to+\infty} \frac{5x}{3x+3} = \lim_{x\to+\infty} \frac{\frac{5x}{x}}{\frac{3x+3}{x}} = \lim_{x\to+\infty} \frac{5}{3+\frac{3}{x}} = \frac{5}{3},$$

since the term $\frac{3}{x}$ gets negligible (very close to 0) as x grows large. Similarly,

$$\lim_{x\to-\infty} \frac{5x}{3x+3} = \lim_{x\to-\infty} \frac{\frac{5x}{x}}{\frac{3x+3}{x}} = \lim_{x\to-\infty} \frac{5}{3+\frac{3}{x}} = \frac{5}{3}.$$

So the only horizontal asymptote is $y = \frac{5}{3}$.
Vertical asymptotes: we check the limit near points where the fnction is not defined. The only such point is $x = -1$, since then the denominator is $3 \cdot (-1) + 3 = 0$, so we need to find $\lim_{x\to-1^-} \frac{5x}{3x+3}$ and $\lim_{x\to-1^+} \frac{5x}{3x+3}$. Plugging in $x = -1.001$, the value of the function is $1668.333\ldots$, plugging in $x = -1.0001$, the value is $16668.333\ldots$, so we may assume $\lim_{x\to-1^-} \frac{5x}{3x+3} = +\infty$. Now, plugging in $x = -0.999$, the value of the function is -1665, plugging in $x = -0.9999$, the value is -16665, so we may assume $\lim_{x\to-1^+} \frac{5x}{3x+3} = -\infty$.
Thus we have one vertical asymptote: $x = -1$.
Due to the two asymptotes we found, we can sketch the function:



(c) $\ln(x^2 + x - 6)$

Solution: As $x \to \pm\infty$, the natural logarithm function grows large (since the range of $\ln(x)$ is $(-\infty, +\infty)$), so we have no horizontal asymptotes. For the vertical asymptotes, we want to first find where the function s defined. Since $x^2 + x - 6 = (x - 2)(x + 3)$, $x^2 + x - 6 > 0$ when either both factors are positive, or both factors are negative.

Case 1: $x - 2 > 0$ and $x + 3 > 0$, so $x > 2$ and $x > -3$, so $x > 2$.

Case 2: $x - 2 < 0$ and $x + 3 < 0$, so $x < 2$ and $x < -3$, so $x < -3$.

Therefore, the function is defined when on the interval $(-\infty, -3) \cup (2, +\infty)$.

We can only have asymptotes at the real endpoints of these intervals, so the vertical asymptotes are $x = 2, x = -3$.