MATH 1 Homework 3 Solutions

Assigned September 28th, due October 5th

1. (a) We start by raising 16 to both sides to get

$$16^{\log_4(x)+\log_{16}(x)} = 16^3.$$

Using exponent rules, we can separate this exponent to get $16^{\log_4(x)} \times 16^{\log_{16}(x)} = 16^3$. We can rewrite $16^{\log_4(x)} = (4^2)^{\log_4(x)} = 4^{2\log_4(x)} = 4^{\log_4(x^2)}$ so that our equation now reads

$$4^{\log_4(x^2)} \times 16^{\log_{16}(x)} = 16^3.$$

But this simplifies to $x^2 \cdot x = 16^3$, so $x^3 = 16^3$, so $x = 16$.

(b) Note that we can rewrite 4 as 2^2, so we can rewrite the entire equation to be

$$2^{(2^x)} = 2^{(2\cdot 4^x)}.$$

Taking $\log_2$ of both sides gives us $2^x = 2 \cdot 4^x$. Again we can take $\log_2$ of both sides to get $\log_2(2^x) = \log_2(2 \cdot 4^x)$. We can separate this out using our log rules to get $x\log_2(2) = \log_2(2) + x\log_2(4)$. Since $\log_2(2) = 1$ and $\log_2(4) = 2$, this is the same as $x = 1 + 2x$, so $x = -1$.

(c) We follow the exact same procedure to get

$$2^{(2^x)} = 2^{(2\cdot 2^x)}.$$

Taking $\log_2$ of both sides gives us $2^x = 2 \cdot 2^x$. Again we can take $\log_2$ of both sides to get $\log_2(2^x) = \log_2(2 \cdot 2^x)$. We can separate this out using our log rules to get $x\log_2(2) = \log_2(2) + x\log_2(2)$. Since $\log_2(2) = 1$, this is the same as $x = 1 + x$. But this is impossible. Thus there is no solution.

(d) First, note that the equation is defined only if $x > 0$, so any solutions we may get should be in the interval $(0, +\infty)$. If we raise 4 to both sides, we get

$$4^{\log_2(x)} = 4^{2\log_4(x)}.$$

Note that we can rewrite 4 as 2^2, so this becomes

$$2^{2\log_2(x)} = 4^{2\log_4(x)}.$$

Using the exponent rule, this becomes $2^{\log_2(x^2)} = 4^{\log_4(x^2)}$. But this cancels to become $x^2 = x^2$. While this is true of all real numbers, we are only looking at solutions in the interval $(0, +\infty)$, so our solutions are all real numbers in the interval $(0, +\infty)$.

2. (a) First, note that the length of AC is $\sqrt{51}$ by the Pythagorean Theorem. Then we can read the trig functions off of the triangle: $\sin(\theta) = \frac{\sqrt{51}}{10}$, $\cos(\theta) = \frac{7}{10}$, and $\tan(\theta) = \frac{\sqrt{51}}{7}$. Based on this, $\frac{\pi}{6} < \theta < \frac{\pi}{3}$ because $\cos(\theta) = .7$, which falls in between the values of $\cos(\pi/6)$ and $\cos(\pi/3)$ (and similarly for sin and tan).

(b) If we put these points on the unit circle, $(\cos(\pi/10), \sin(\pi/10))$ would be just above the x axis in quadrant 1, $(\cos(11\pi/10), \sin(11\pi/10))$ would be just under the x axis in quadrant 3, and $(\cos(14\pi/10), \sin(14\pi/10))$ would be just to the left of the y axis in quadrant 3. Based on the picture, $\cos(14\pi/10) > -\frac{1}{2}$ because the if we draw a straight line up to the x axis, it hits the x axis at a point to the right of $-\frac{1}{2}$. Based on the picture, $\sin(14\pi/10) < -\frac{1}{2}$ because if we draw a straight line over to the y axis, we hit the y axis at a point below $-\frac{1}{2}$.

3. (a) The formula for the new period is $\frac{2\pi}{\omega}$, where ω is the horizontal scaling factor. Thus, the period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$.

(b) The period for $\cos(3x)$ is $\frac{2\pi}{3}$, so we can get the whole graph for $\cos(3x)$ by putting it together with either strips of length $\frac{2\pi}{3}$, or multiples of these of lengths $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \frac{8\pi}{3}, \ldots$ (so the function will also repeat itself after these periods of time). On the other hand, the period for $\tan(2x)$ is $\frac{\pi}{2}$, so we can get the whole graph for $\tan(2x)$ by putting it together with either strips of length $\frac{\pi}{2}$, or multiples of these of lengths $\frac{\pi}{2}, \frac{2\pi}{2}, \frac{3\pi}{2}, \frac{4\pi}{2}, \ldots$. Among the two lists we see that both functions will repeat themselves after $\frac{6\pi}{3} = 2\pi = \frac{4\pi}{2}$, so their sum will also repeat itself after 2π. Thus, a period for the sum is 2π. Alternatively, we can check this period works algebraically:

$$\cos(3(x + 2\pi)) + \tan(2(x + 2\pi)) = \cos(3x + 6\pi)) + \tan(2x + 4\pi)) = \cos(3x) + \tan(2x).$$

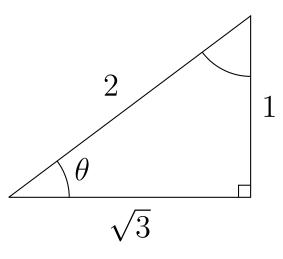
(c) We proceed as in the previous exercise: $\sin(\frac{1}{4}x)$ has period 8π, so it will repeat itself after $8\pi, 16\pi, 24\pi, 32\pi, 40\pi, 48\pi, \ldots$. On the other hand, $\cos(\frac{1}{5}x)$ has period 10π, so it will repeat itself after $10\pi, 20\pi, 30\pi, 40\pi, 50\pi, \ldots$. Both functions repeat themselves after 40π, and so will their product, so a period for the product is 40π. Alternatively, we can check this period works algebraically:

$$\sin\left(\frac{1}{4}(x + 40\pi)\right)\cos\left(\frac{1}{5}(x + 40\pi)\right) = \sin\left(\frac{1}{4}x + 10\pi\right)\cos\left(\frac{1}{5}x + 8\pi\right) = \sin\left(\frac{1}{4}x\right)\cos\left(\frac{1}{5}x\right).$$

4. (a) The range of arcsin is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so we look at an angle θ in this range on the unit circle that has $\sin(\theta) = -\frac{\sqrt{3}}{2}$. This angle is $-\frac{\pi}{3}$.

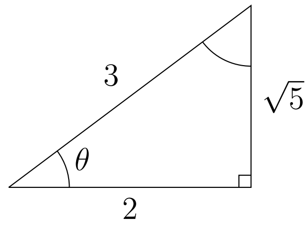
(b) We check that $\cos(\frac{3\pi}{2}) = 0$, so $\arccos(\cos(\frac{3\pi}{2})) = \arccos(0) = \frac{\pi}{2}$. We cannot use the cancellation laws for inverse functions since we need our angle to be in the interval $[0, \pi]$.

(c) We have the following triangle:



The third side length is given by the Pythagorean Theorem : $\sqrt{2^2 - 1^2} = \sqrt{3}$. The angle we are interested in is θ, and $\cos(\theta) = \frac{\sqrt{3}}{2}$.

(d)



The third side length is given by the Pythagorean Theorem : $\sqrt{3^2 - 2^2} = \sqrt{5}$. The angle we are interested in is θ, and $\tan(\theta) = \frac{\sqrt{5}}{2}$.

5. Let $f(x) = e^{\sin(x)}$. To find an inverse, we first restrict to a one-to-one domain. Since the exponential function in base e is an increasing function, it is one-to-one when the exponent $\sin(x)$ is one-to-one. We choose the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then the range will be $[e^{\sin(-\pi/2)}, e^{\sin(\pi/2)}]$ which is $[\frac{1}{e}, e]$.

   Next, we find the inverse:

$$y = e^{\sin(x)}$$
$$\ln(y) = \sin(x)$$
$$\arcsin(\ln(y)) = x.$$

   Now we relabel, and get $f^{-1}(x) = \arcsin(\ln(x))$. Since for inverses the domains and range get flipped from the original function, the domain for f^{-1} is $[\frac{1}{e}, e]$ and the range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.