MATH 1 Homework 2

Assigned September 21st, due September 28th

1. (a) You're riding a bike and want to see how fast you can go. After 2 seconds you've gone 1 meter, after 4 seconds you've gone 5 meters, and after 10 seconds you've gone 20 meters. Use Lagrange interpolation to find a polynomial that fits these data points, and then use that polynomial to predict how far you will have gone in 15 seconds. (Leave the polynomial unsimplified and use a calculator to evaluate).

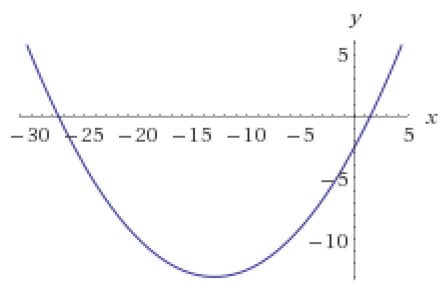
   Solution: My three points are $(2, 1)$, $(4, 5)$, and $(10, 20)$. Plugging these into my formula gives me

$$f(x) = 1\frac{(x-4)(x-10)}{(2-4)(2-10)} + 5\frac{(x-2)(x-10)}{(4-2)(4-10)} + 20\frac{(x-2)(x-4)}{(10-2)(10-4)}.$$

   When I plug 15 into this, I get $f(15) = 35.94$.

   (b) Use a graphing tool to find the plot of your polynomial that you got in part (a). Sketch the plot. Does this model make sense? Why or why not?

   Solution:



   This model does not make sense because after 0 seconds, I will have gone a negative distance.

2. Let f be a function with domain $[2, 3]$ and range $[-1, 2]$. Give the domain and range of the following functions:

   (a) $f(3x + 2) + 3$

   Solution: For domain: To be able to plug $3x + 2$ into f, it needs to be in the domain, so $2 \leq 3x + 2 \leq 3$. Now we need to solve for x, so subtracting 2 gives $0 \leq 3x \leq 1$, and dividing by 3 gives $0 \leq x \leq \frac{1}{3}$. Thus our final domain is $[0, 1/3]$.

   For range: Just add 3: $[2, 5]$.

   (b) $4f(-x + 1) - 1$

   Solution: For domain: To be able to plug $-x + 1$ into f, it needs to be in the domain, so $2 \leq -x + 1 \leq 3$. Now we need to solve for x, so subtracting 1 gives $1 \leq -x \leq 2$, and dividing by -1 gives $-1 \geq x \geq -2$. Thus our final domain is $[-2, -1]$.

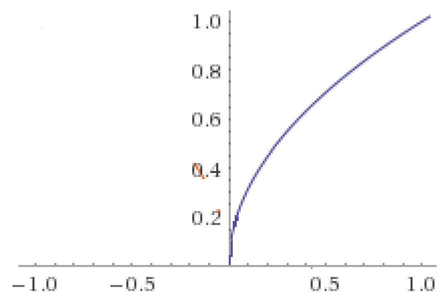
   For range: We first multiply by 4 to get $[-4, 8]$, and then subtract 1, giving us $[-5, 7]$.

(c) $-2f(x+1)+1$

Solution: For domain: To be able to plug $x+1$ into f, it needs to be in the domain, so $2 \leq x+1 \leq 3$. Now we need to solve for x, so subtracting 1 gives $1 \leq x \leq 2$. Thus our final domain is $[1,2]$.

For range: We first multiply by -2 and switch the order of the bounds to get $[-4,2]$. Then we add 1 to get $[-3,3]$.

3. Below is the graph of a function f:

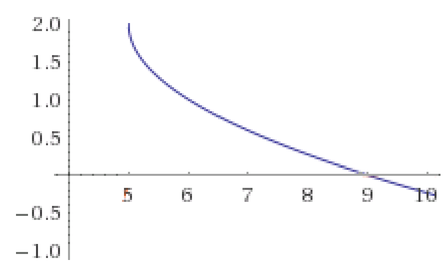


(a) Describe the transformations needed to draw the graph of $-f(x-5)+2$. What order do the transformations need to be performed in?

Solution: The -1 reflects over the x axis, the -5 moves the graph 5 units to the right, and the $+2$ moves the graph 2 units up. The reflection must happen before moving the graph 2 units up, and the -5 can happen at any time.

(b) Sketch the graph of $-f(x-5)+2$.

Solution:



(c) Why does the order matter? What would happen if you did the transformations in a different order?

Solution: If we swapped the order of reflection and moving up 2 units, the graph would start at $y=-2$ instead of $y=2$.

4. For the following functions, decompose into the given number of nonidentity functions.

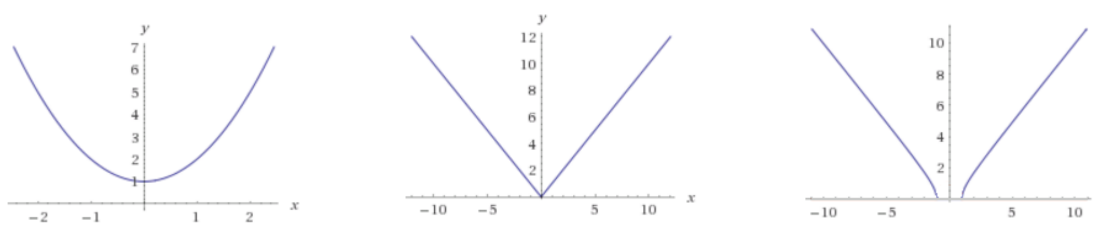
   (a) Write $\sqrt[3]{x^2+4x+4}$ as the composition of two nonidentity functions.
   Solution: We can set $f(x) = x^2+4x+4$, $g(x) = \sqrt[3]{x}$, and $(g \circ f)(x) = \sqrt[3]{x^2+4x+4}$.

   (b) Write $\sqrt[3]{x^2+4x+4}$ as the composition of three nonidentity functions.
   Solution: We notice furthermore that as defined above, $f(x) = x^2+4x+4 = (x+2)(x+2) = (x+2)^2$, so if we let $h(x) = (x+2)$, $j(x) = x^2$, then $f = j \circ h$. Then with $g(x) = \sqrt[3]{x}$, we have
   $$(g \circ j \circ h)(x) = \sqrt[3]{(x+2)^2}$$

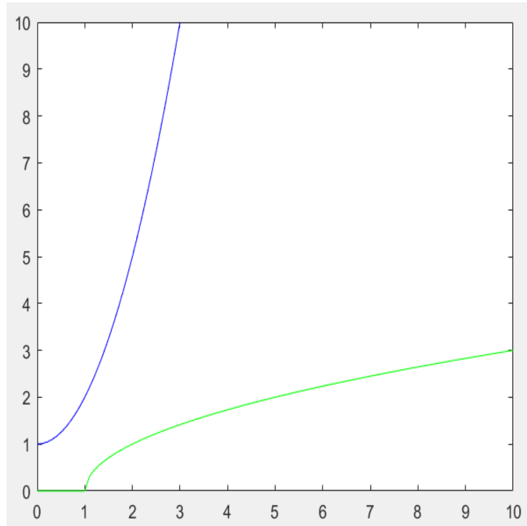
   (c) Write $\sqrt{x} + 34x^2$ as the composition of two nonidentity functions.
   Solution: $\sqrt{x} + 34x^2 = \sqrt{x} + 34(\sqrt{x})^4$,
   so let $g(x) = \sqrt{x}$, $h(x) = x + 34x^4$. Then $(h \circ g)(x) = \sqrt{x} + 34(\sqrt{x})^4$.

   (d) Write $-f(x-5)+2$ as the composition of three nonidentity functions.
   Solution: Let $g(x) = x-5$, $h(x) = -x+2$. Then
   $$(h \circ f \circ g)(x) = h(f(g(x))) = h(f(x-5)) = -f(x-5)+2.$$

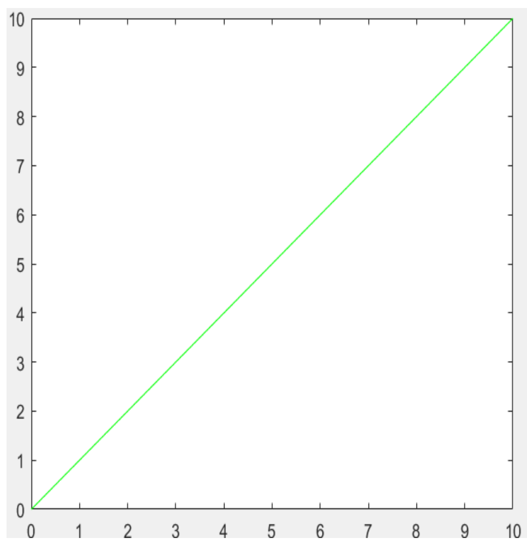
5. (a) Sketch the graphs of the following functions
   $$f(x) = x^2+1, \quad g(x) = |x|, \quad h(x) = \sqrt{x^2-1}.$$
   Solution: From left to right we have f, g, h



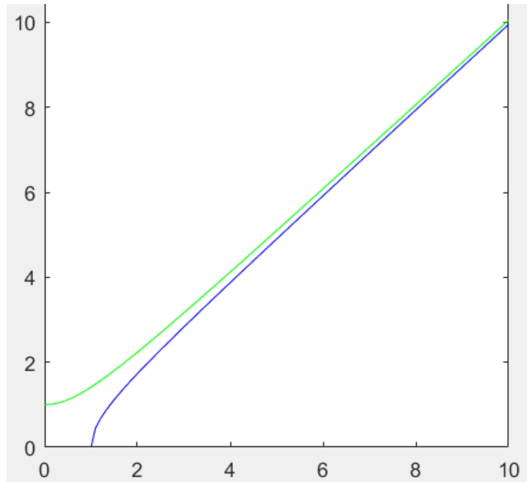
   (b) For each function you have sketched, choose a maximal part of the domain where the function is one-to-one *(by maximal we mean that if we add another point, the function wouldn't be 1-1)*. Specify this part of the domain, then sketch the inverse function with respect to that domain.
   Solution: f is not one-to-one on the whole real line, but it is one-to-one on $[0, +\infty)$. Its inverse is symmetric with respect to the $y = x$ line, and is depicted in green in the following figure



g is also not one-to-one on the whole real line, but it is one-to-one on $[0, +\infty)$. Its inverse is symmetric with respect to the $y = x$ line, and since the graph of g for $x \geq 0$ **is** in the line $y = x$, its reflection across the line is also on the line, so the inverse of g with respect to the domain $[0, +\infty)$ has the same graph as g itself from $[0, +\infty)$, and the graph of the inverse is depicted in green in the following figure:



h is not one-to-one on the whole real line either, but it is one-to-one on $[1, +\infty)$. Its inverse is symmetric with respect to the $y = x$ line, and is depicted in green in the following figure



(c) Find the equation of the inverse for each function.

Solution: The equations of the inverses are found the following way:

For $f(x)$:

$$y = x^2 + 1$$
$$y - 1 = x^2$$
$$\sqrt{y-1} = x$$

so $f^{-1}(y) = \sqrt{y-1}$ and after relabeling, $f^{-1}(x) = \sqrt{x-1}$.

For $g(x) = |x|$, we have that $g(x) = x$ is already the identity when $x \geq 0$, so the inverse is $g^{-1}(x) = x$.

For $h(x)$:

$$y = \sqrt{x^2 - 1}$$
$$y^2 = x^2 - 1$$
$$y^2 + 1 = x^2$$
$$\sqrt{y^2+1} = x$$

so $h^{-1}(y) = \sqrt{y^2+1}$ and after relabeling, $h^{-1}(x) = \sqrt{x^2+1}$.

6. Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer? Explain your answer.

- Seven million dollars at the end of the month.
- One cent on the first day of the month, two cents on the second day, four cents on the third day, and, in general, 2^{n-1} cents on the nth day. (This month has 30 days).

Solution: The second choice: on the 29th day we get $2^{29-1} = 2^{28}$ cents, which is 268435456 cents, a bit more than 2684354 dollars; on the 30th day we get $2^{30-1} = 2^{29}$ cents, which is 536870912 cents, and therefore a bit more than 5368709 dollars. In the last two days alone, we make $2684354 + 5368709 > 7000000$ dollars.