MATH 1 HW 1 Solutions

1. $f(x) = -x^2 + 1$, which is defined on all real numbers, so the domain is $(-\infty, +\infty)$. Since $-x^2$ can be at most 0, and otherwise it is negative, the range is $(-\infty, 1]$.

   $g(x) = 2x$ is defined on all real numbers, so the domain is $(-\infty, +\infty)$. $g(x)$ can be any real number a when $x = \frac{a}{2}$: $g(\frac{a}{2}) = 2\frac{a}{2} = a$, so the range is all real numbers $(-\infty, +\infty)$.

   $h(x) = \sqrt{x}$. Because of the square root, h is defined only when $x \geq 0$, so the domain is $[0, +\infty)$. $h(x)$ can take any nonnegative number $b \geq 0$ when $x = b^2$: $h(b^2) = \sqrt{b^2} = b$, so the range is $[0, +\infty)$.

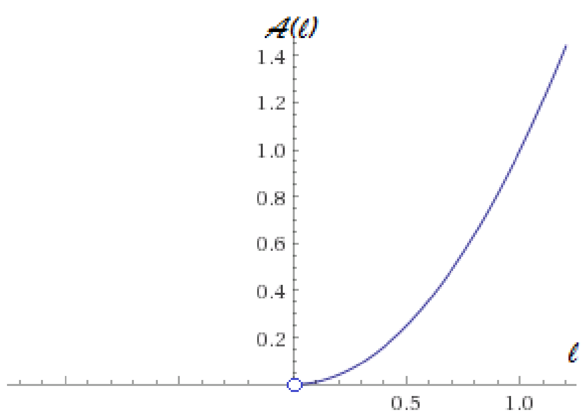
   $(g + h)(x) = 2x + \sqrt{x}$. Since we can only plug in values of x that are both in the domain of g and the domain of h, the domain for $g + h$ is $[0, +\infty)$. For $x \geq 0$, both $g(x)$ and $h(x)$ can be any nonnegative number, so their sum can be any nonnegative number as well, and therefore the range of $g + h$ is $[0, +\infty)$.

   $\dfrac{1}{g}(x) = \dfrac{1}{2x}$. We can't divide by 0, so the domain is $(-\infty, 0) \cup (0, +\infty)$. $\frac{1}{g}(x)$ can be any nonzero number $a \neq 0$ when $x = \dfrac{1}{2a}$: $\dfrac{1}{g}(\dfrac{1}{2a}) = \dfrac{1}{2\frac{1}{2a}} = a$, so the range is $(-\infty, 0) \cup (0, \infty)$.

   $(f \circ h)(x) = f(h(x)) = -(\sqrt{x})^2 + 1 = -x + 1$. Although this line in general is defined everywhere, since in our composition we did h first, we started in the domain of h, so x must be in the domain of h to begin with! Therefore, the domain can be at most $[0, \infty)$. Now, since the range $[0, +\infty)$ of h is contained in the domain $(-\infty, +\infty)$ of f, we actually have that the domain of $f \circ h$ is the entire domain of h, that is, $[0, +\infty)$. Therefore, our line starts at $x = 0$, so the range is $(-\infty, 1]$.

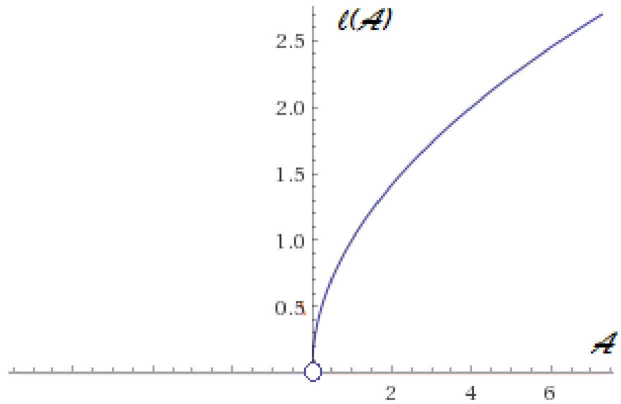
   $(h \circ f)(x) = h(f(x)) = \sqrt{-x^2 + 1}$. In this composition, we did f first, so we start in the domain of f which is $(-\infty, \infty)$. However, only a part of the range of f is in the domain of h, which is where $-x^2 + 1 \geq 0$, that is, when $x^2 \leq 1$ and therefore x is in the interval $[-1, 1]$. Therefore, the domain of $h \circ f$ is the interval $[-1, 1]$. Since $-x^2 + 1$ on the interval $[-1, 1]$ has range $[0, 1]$, the square root has the same range $[0, 1]$.

2. (a) If the length of the side of a square is l, then the area is $A(l) = l^2$. We can only have positive lengths, so the domain of l is $(0, +\infty)$, and the graph looks the following:



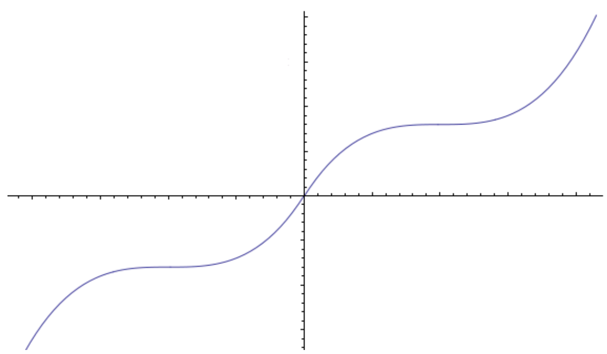
   Three points on the graph are $(1, 1), (2, 4), (3, 9)$.

(b) If the area of a square is A, then the length of its side is $l(A) = \sqrt{A}$. We can only have positive areas, so the domain is $(0, +\infty)$, and the graph looks the following:

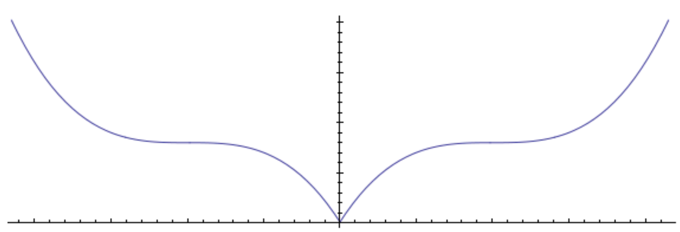


Three points on the graph are $(1, 1), (4, 2), (9, 3)$.

3. (a) If f is odd, it is symmetric with respect to the origin, so if we look at the graph upside-down, it should look the same. Thus the graph of f should look something like this:



(b) If f is even, it must be symmetrical with respect to the y-axis, so if we fold the graph along the y-axis, the branches of the graph would coincide. Thus, the graph of f should look like this:



(c) No: if h is increasing and $b \neq 0$, then either $b > -b$ or $b < -b$. In any case, either $h(b) > h(-b)$ or $h(b) < h(-b)$, so h can't be even. It can however be both incresing and odd: let $h(x) = x^3$. Then h is increasing, and $h(-x) = (-x)^3 = -x^3 = -h(x)$, so h is odd.

4. (a) $\{a_n\} = \{n\}$ is increasing but not bounded. $\{b_n\} = \{\frac{1}{n}\}$ is decreasing and bounded by 1 (or 2 or 3, etc), since the largest value that it can have is when $n = 1$. $\{c_n\} = \{\frac{a_n}{b_n}\} = \{n^2\}$ is increasing but not bounded. $\{d_n\} = \{(-1)^n\}$ is neither increasing nor decreasing. It is bounded by 1 (or 2 or 3, etc).

   (b) If both $\{l_n\}$ and $\{p_n\}$ are increasing, $\{q_n\}$ can also be increasing. For example, if $\{p_n\} = \{n^2\}$ and $\{q_n\} = \{n\}$, then $\{l_n\} = \{n\}$. $\{q_n\}$ can also be decreasing: if $\{p_n\} = \{n\}$ and $\{q_n\} = \{\frac{1}{n}\}$, then $\{l_n\} = \{n^2\}$.

   If $\{p_n\}$ is increasing and $\{l_n\}$ is decreasing, then $\{q_n\}$ can be increasing. For example, if $\{p_n\} = \{n\}$ and $\{q_n\} = \{n^2\}$, then $\{l_n\} = \{\frac{1}{n}\}$. $\{q_n\}$ cannot be decreasing: If $\{q_n\}$ is decreasing, then since it is in the denominator, and the numerator is increasing, this causes the entire sequence to increase.

5. (a) $\frac{f(2.5)-f(2)}{2.5-2} = \frac{0-16}{.5} = -32$

   (b) $\frac{f(2.1)-f(2)}{2.1-2} = \frac{13.44-16}{.1} = -25.6$

   (c) $\frac{f(2.05)-f(2)}{2.05-2} = \frac{14.76-16}{.05} = -24.8$

   (d) $\frac{f(2.01)-f(2)}{2.01-2} = \frac{15.7584-16}{.01} = -24.16$

   The answers are getting closer to -24.

6. (a) This is not constant or linear because it is not a straight line. It is not a power function because it does not pass through the origin.

   (b) It is not constant or linear because it is not a straight line.

   (c) This is not constant or linear because it is not a straight line. It is not a power function because it does not pass through the origin. It is not a polynomial because it's domain is not all real numbers. It is not a rational function because there is an interval missing from its domain (a rational function only has single points missing from its domain).