# Quadratic Functions

## Quadratic Functions

A quadratic function is a function of the form $f(x) = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants and $a \neq 0$. The term $ax^2$ is called the quadratic term (hence the name given to the function), the term $bx$ is called the linear term, and the term $c$ is called the constant term.

Quadratic functions are very good for describing the position of particles under constant (or near constant) acceleration. The acceleration due to gravity of a planet or moon near its surface is an example of near constant acceleration. Suppose that twenty years from now you are on a mission to the Moon. You just finished your excursion on the surface, you are ascending back to the lunar orbiter in your lander at a velocity of $V$ meters per second, when, suddenly, the lander starts breaking up. At a height of $H$ meters, you decide that your only chance to survive is to jump out of the lander and hope for the best. The acceleration due of gravity near the surface of the Moon is $A$ meters per second$^2$, and there is little or no atmosphere on the Moon, hence little or no air resistance. If $t$ is the time in seconds since you began your dive, then your height $h(t)$ as a function of $t$ is

$$h(t) = -\frac{1}{2}At^2 + Vt + H.$$

On the way down, you realize that it may not have been the brightest idea to jump out of the lander, and you decide to figure the exact second when you will become a human pancake. So you want to know how to solve the equation $h(t) = 0$ for $t$. More generally, we ask: suppose $f(x) = ax^2 + bx + c$ and we want to know for what values of $x$ does $f(x) = 0$. The most complete answer is the quadratic formula, as it gives an exact answer for any quadratic function you can think of:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Notice that if $b^2 - 4ac > 0$ , then $f(x)$ has two real roots, if $b^2 - 4ac = 0$, then $f(x)$ has only one real root, and if $b^2 - 4ac < 0$, then $f(x)$ has two complex roots. The quantity $b^2 - 4ac$ is called the discriminant of the quadratic function. We strongly urge you to memorize the quadratic formula.

## Graphs of Quadratic Functions

Let us graph the quadratic function $f(x) = x^2 - 4x + 3$. First, we need a numerical table of values:

| $x$ | $f(x)$ |
|---|---|
| -1 | 8 |
| 0 | 3 |
| 1 | 0 |
| 2 | -1 |
| 3 | 0 |
| 4 | 3 |
| 5 | 8 |

If we plot out the points we get from this numerical table, and we smoothly connect the points, we get a U-shaped curve called a parabola. The graphs of all quadratic functions are parabolas.

A very important characteristic of all parabolas is that they have an axis of symmetry, meaning that for every parabola there is a vertical line that we can draw through that parabola for which the part of that parabola left of that line is a mirror image of the part of that parabola right of that line. For the graph of the function $f(x) = x^2 - 4x + 3$, that axis of symmetry is the line $x = 2$. For the general quadratic function $f(x) = ax^2 + bx + c$, the axis of symmetry is the line $x = -\frac{b}{2a}$.

Next let us consider the graph of the function $g(x) = ax^2$ as we change $a$. Let us start with $a = 1$ and make $a$ more positive. The parabola which is the graph of $g(x) = ax^2$ becomes narrower as a result. What if make $a$ less positive, but keep $a > 0$? Then we find that the parabola becomes wider. In both cases, the parabola intersects the $y$-axis at the same point, the origin. What if we try $a = -1$? Then we get the same shape parabola as we got for the graph of $g(x) = x^2$, but now it is open downward. This is an important characteristic of the graphs of all quadratic functions: if $a > 0$, then the graph of $f(x) = ax^2 + bx + c$ is a parabola open upward, and if $a < 0$, then it is a parabola open downward. Finally, if we make $a$ more negative than $-1$, then we get a more narrow parabola open downward, and if we make $a$ less negative then $-1$ but still less than 0, then we get a less narrow parabola open downward. Again, for all $a \neq 0$, the graph of $g(x) = ax^2$ intersects the $y$-axis at the origin.

Continuing with our theme, let us consider the graph of the function $h(x) = x^2 + bx$ as we change $b$. We start by drawing the graph when $b = 0$, that is, the graph of $h(x) = x^2$. We now draw the graph of $h(x) = x^2 + 2x$. We get a parabola which has the same shape and which is pointed upward, and which intersects the $y$-axis at the origin, but for which the axis of symmetry is now the line $x = -1$. Some of the parabola has dipped below the $x$-axis, so now the function has two roots instead of one. If we try graphing $h(x) = x^2 + 4x$, we get a result along the same lines: same shape, pointed upward, same $y$-intercept, but now the axis of symmetry is $x = -2$, and much more of the graph is below the $x$-axis. Suppose now we try graphing $h(x) = x^2 - 2x$. We get the mirror image of the graph we got for $x^2 + 2x$: same shape, pointed upward, same $y$-intercept, but with an axis of symmetry of $x = 2$. To complete the picture, we graph $h(x) = x^2 - 4x$, and we find that its graph is the mirror image of the graph of $x^2 - 4x$. You can imagine that the action of changing $b$ in effect shifts the graph of the quadratic function without changing its shape, direction, or $y$-intercept, as if the parabola is rigid and permanently fixed to the $y$-axis at its intercept.

Finally, if we graphed the function $k(x) = x^2 + c$ for various values of $c$, we would find that the $y$-intercept of the graph would change, but nothing else, just the same as for linear and constant functions.

Now, we notice that, when $a$ is positive, the graph of $f(x) = ax^2 + bx + c$ has a lowest point. That lowest point is exactly where the axis of symmetry of the parabola intersects the parabola. The value of the function $f(x)$ at that point, in other words, $f(-\frac{b}{2a})$, is called the minimum of the function: the value of the function can never be any lower than $f(-\frac{b}{2a})$. For example, the minimum of the function $f(x) = x^2 - 4x + 3$ is equal to

$$f(2) = 2^2 - 4 \cdot 2 + 3 = 4 - 8 + 3 = -1.$$

Likewise, if $a$ is negative, the graph of $f(x) = ax^2 + bx + c$ has a highest point, since it is open downward, and that highest point is exactly where the axis of symmetry crosses the parabola. The value of the function at that highest point, which again is $f(-\frac{b}{2a})$, is called the maximum of $f(x)$: the value of $f(x)$ can never be any larger than $f(-\frac{b}{2a})$. For example, the function $g(t) = -3t^2 + 6t + 2$ has a maximum at $x = -\frac{6}{-6} = 1$, and that maximum is

$$g(1) = -3 \cdot 1^2 + 6 \cdot 1 + 2 = -3 + 6 + 2 = 5.$$

To summarize, every quadratic function has either a minimum or a maximum. The quadratic function $f(x) = ax^2 + bx + c$ has a minimum if $a > 0$ and a maximum if $a < 0$. If both cases, the value of that minimum is $f(-\frac{b}{2a})$.

## Tangent Lines

We now begin the study of our first calculus technique. Consider the graph of the quadratic function $f(x) = ax^2 + bx + c$. At every point along the parabola, we draw exactly one line which intersects the parabola precisely at that point, but does not cross the parabola at that point. That unique line is called the tangent line of the graph at that point.

Specifically, consider the tangent line to the graph of $f(x) = ax^2 + bx + c$ at the point $(p, f(p))$. There is a special name for the slope of that line: it is called the derivative of $f(x)$ at $x = p$. There are two standard notations for the derivative of $f(x)$ at $x = p$:

$$f'(p) \qquad \text{and} \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(p).$$

You may use either of these notations, but the second is recommended.

Given the quadratic function $f(x) = ax^2 + bx + c$, there is a algebraic formula for the derivative of $f(x)$ at $p$:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(p) = 2ap + b.$$

So, in other words, the slope of the tangent line to the graph of $f(x)$ at the point $(p, f(p))$ is $2ap + b$. With this algebraic formula in hand, we can find the formula for the tangent line to the graph of $f(x)$ at any point.

For example, let $f(x) = x^2 - 4x + 3$. Let us find the formula for the tangent line to $f(x)$ at $x = 3$. First, we need to know the point $(3, f(3))$. Earlier, we found that $f(3) = 0$, so the point on the parabola through which the tangent line will pass is $(3, 0)$. Second, we need to know the slope of the tangent line at $x = 3$. Applying our formula, we take $a = 1$, $b = -4$, and $p = 3$:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(p) = 2ap + b = 2 \cdot 1 \cdot 3 + (-4) = 6 - 4 = 2.$$

So we know the slope of the tangent line is 2 and that it passes through the point $(3, 0)$. We can know figure out a formula for the tangent line using the techniques we learned during the last class:

$$\begin{aligned} y &= mx + b \\ 0 &= 2 \cdot 3 + b \\ 0 &= 6 + b \\ -6 &= b. \end{aligned}$$

Therefore the equation for the tangent line to the graph of $f(x) = x^2 - 4x + 3$ at $x = 3$ is $y = 2x - 6$.

For another example, let us derive the equation for the tangent line to the graph of $g(t) = -3t^2 + 6t + 2$ at $t = 1$. Recall that $g(t)$ achieves its maximum at $t = 1$, at that maximum is $g(1) = 5$. Let us compute the derivative of $g(t)$ at $t = 1$: setting $a = -3$, $b = 6$, and $p = 1$, we get

$$\frac{\mathrm{d}g}{\mathrm{d}t}(p) = 2ap + b = 2 \cdot (-3) \cdot 1 + 6 = -6 + 6 = 0.$$

(Notice the change in notation for the derivative to reflect that the function is $g$ and the independent variable is $t$.) We now know that the tangent line to the graph of $g(t)$ at $t = 1$ has slope 0 and passes through the point $(1, 16)$. We can therefore find the equation for the line:

$$\begin{aligned} y &= mt + b \\ 5 &= 0 \cdot 1 + b \\ 5 &= 0 + b \\ 5 &= b. \end{aligned}$$

Thus the equation for the tangent line is $y = 5$.

Here is a question for you to think about for next class: does it make sense that the derivative of $g(t)$ at $t = 1$, at the point where it achieves its maximum, would be 0. More importantly, could it be any number but 0? The answer to this question tells us why the derivative is such an important tool.