Inverse Functions

Defining Inverse Functions

In our final lecture of the term, we discuss the concept of the inverse of a function. Let $f(x)$ be a real-valued function. We say that $g(x)$ is the inverse of $f(x)$ if for all values of x in the domain of $f(x)$, we have that

$$g(f(x)) = x,$$

and for all values of x in the domain of $g(x)$, we have that

$$f(g(x)) = x.$$

So composing g after f cancels out the action of f, and composing f after g cancels out the action of g. We usually denote the inverse of $f(x)$, if it exists, by $f^{-1}(x)$. If $f(x)$ is the inverse of $g(x)$, then $g(x)$ is also the inverse of $f(x)$.

We already have an important example of inverse functions: the inverse of the exponential function e^x is the natural logarithmic function $\ln x$, since for all real numbers x, we have that

$$\ln(\mathrm{e}^x) = x,$$

and for all values of x in the domain of $\ln x$ (what is this domain?) we get that

$$\mathrm{e}^{\ln x} = x.$$

Likewise, the functions a^x and $\log_a x$ are inverses to each other, because we defined $\log_a x$ specifically to have this property.

Another example of inverse functions are x^n when n is a odd natural number and the nth root function $\sqrt[n]{x} = x^{\frac{1}{n}}$. Both of these functions are defined everywhere, and we have that

$$\sqrt[n]{x^n} = (x^n)^{\frac{1}{n}} = x^{n \cdot \frac{1}{n}} = x^1 = x = x^1 = x^{\frac{1}{n} \cdot n} = \left(x^{\frac{1}{n}}\right)^n = \left(\sqrt[n]{x}\right)^n.$$

This concept of inverse function gets a little more tricky when we consider the function $\sqrt[n]{x}$ when n is an even natural number (like the usual square root function). The function $\sqrt[n]{x} = x^{\frac{1}{n}}$ is always non-negative (that is, positive or 0). This restriction makes an impact on the domain of the inverse function: we would guess that x^n would be the inverse to $\sqrt[n]{x}$, and this is true, but x^n is defined for all real numbers. If x is a negative number, then

$$\sqrt[n]{x^n} \neq x,$$

because x is negative, but the nth root of any number is non-negative when n is an even number. Thus $\sqrt[n]{x}$ does not cancel out the action of x^n when x is a negative number. The solution is to restrict the domain of x^n to non-negative numbers, and, indeed, x^n on the domain $x \geq 0$ is the inverse function of $\sqrt[n]{x}$. The lesson here is that we always need to be conscious of the domain of a function as well as its formula, because studying the formula alone can give us the wrong answer.

Some functions do not have inverse functions: for these functions, there is no way to cancel out their action on x, to put the machine into reverse, once that action is done. Chief among these functions are the constant functions $f(x) = c$. A constant function sends every number x to some constant c. If $g(x)$ were the inverse to $f(x) = c$, then for every value of x we would have that

$$x = g(f(x)) = g(c).$$

So $g(c)$ would have to have every real number as a value, which is impossible, because as a function, $g(c)$ can take exactly one value and one value only. So constant functions cannot have inverse functions. Likewise, consider the function $f(x) = x^2$ defined everywhere. If $g(x)$ is the inverse of $f(x) = x^2$, then, specifically, for $x = 1$ and $x = -1$ we would have

$$1 = g(f(1)) = g(1^2) = g(1) = g((-1)^2) = g(f(-1)) = -1.$$

Thus if $g(x)$ is the inverse function of $f(x) = x^2$ (defined everywhere), then both $g(1) = 1$ and $g(1) = -1$, which is impossible if $g(x)$ is indeed a function. So $f(x) = x^2$ defined everywhere does not have an inverse. We can restrict its domain, however, as we saw in the previous paragraph, and get a function which does have an inverse. Why does $f(x) = x^2$ defined everywhere not have an inverse, while $f(x) = x^2$ defined on the non-negative numbers does? We examine their graphs to find out.

Graphs of Inverse Functions and the Horizontal Line Test

The graph of a function and the graph of its inverse are related. To see this relationship, plot the graph of e^x and the graph of $\ln x$ on the same set of axes. Now plot the line $y = x$. What you should see is that the graph of e^x and the graph of $\ln x$ are exact reflections of each other in the line $y = x$. This is true not just for e^x and $\ln x$, but for the graphs of $f(x)$ and $f^{-1}(x)$ as well: the graph of $f^{-1}(x)$, when this function exists, is the reflection of the graph of $f(x)$ in the line $y = x$.

Why is this the case? When we reflect the point (x, y) in the line $y = x$, we get the point (y, x) (test this on a few points to convince yourself of this). Every point on the graph of $f(x)$ is of the form $(x, f(x))$, so every point in the reflection of the graph of $f(x)$ in the line $y = x$ is of the form $(f(x), x)$. If the reflection is the graph of a function, then this function takes $f(x)$ and sends it to x for every x in the domain of f, which is precisely what the inverse function is supposed to do.

Thus, knowing what the graph of x^3 looks like, you should be able to draw the graph of $\sqrt[3]{x}$, and knowing how to draw the graph of x^2 for $x \geq 0$, you should be able to sketch the graph of $\sqrt{x}$.

What about the constant function $f(x) = c$? The graph of this function is a horizontal line. Therefore, if we reflect the graph of $f(x) = c$ in the line $y = x$, we get the vertical line $x = c$. This vertical line cannot be the graph of a function, since it obviously does not pass the vertical line test. So this graphically justifies why we say that constant functions do not have inverses.

What about the function $f(x) = x^2$ defined everywhere? When we reflect it in the line $y = x$, the result is the solution curve of the relation $x = y^2$, which is a parabola on its side. If you sketch this solution curve, you see that it does not pass the vertical line test either, particularly at $x = 1$, as we showed before algebraically. So $f(x) = x^2$ defined everywhere does not have an inverse either.

These two examples give us a clue as to a test we can use to see it a function does not have an inverse. In both of these cases, the reflection of the graph of the function failed the vertical line test: some vertical line $x = a$ passed through the reflection at two points or more. What does this tell us about the graph of the original function? Try reflecting back the vertical line $x = a$: you get the horizontal line $y = a$. Superimpose this horizontal line onto the graph of the original function. You should see that it crosses the graph of the original function at two or more points. So, if the reflection of the graph of a function fails the vertical line test, then some horizontal line passes through the graph of the function more than once, and vice versa. This is the horizontal line test: if $f(x)$ is some real-valued function and there exist some horizontal line $y = a$ which passes through the graph of $f(x)$ more than once (in other words, $f(x) = a$ for more than one value of x), then $f(x)$ does not have an inverse. Thus $f(x) = x^2$ defined everywhere does not have an inverse because its graph, a parabola open upward, certainly does not pass the horizontal line test. The graph of the absolute value function does not pass the horizontal line test, and neither does any of the trigonometric functions, so none of these functions have inverses. If we restrict these functions to smaller domains, however, as we did by restricting $f(x) = x^2$ to the non-negative numbers, then these function may have inverses on these smaller domains. This is, for example, how we define the inverses of the trigonometric functions, which you will study in detail next term.