# General Power Functions

## Defining General Power Functions

In previous lectures, we discussed the properties and derivatives of positive power functions and negative power functions. Today, we discuss power functions in general.

A power function is a function of the form

$$f(x) = x^a,$$

where $a$ is any real number. We understand intuitively what it means to raise $x$ to the power of a natural number $n$: we just multiply $n$ copies of $x$ together. We know what it means to raise $x$ to the $-n$ power: just divide 1 by $x^n$. Now we are asked to raise $x$ to the power of $a$, where $a$ is any real number, but what does, say, $3^\pi$ mean? It does not make sense to multiply $\pi$ copies of 3 together. We have been ignoring this question since we defined exponential functions, but now we intend to give you an answer.

First, restrict $x$ to the positive real numbers and consider the exponential function $\mathrm{e}^x$ and the natural logarithmic function $\ln x$. There is a great relationship built into $\mathrm{e}^x$ and $\ln x$: if we compose these two functions with each other in either order, the resulting function is $x$:

$$\mathrm{e}^{\ln x} = x \qquad \text{and} \qquad \ln(\mathrm{e}^x) = x.$$

This relationship is built into the definition of the natural logarithmic function: the quantity $\ln x$ is defined to be the number such that e raised to the power of that number is $x$. Thus we get the first equation above automatically. The second equation comes from asking the question: to what number must we raise e to get $\mathrm{e}^x$? Obviously, the answer to this question is $x$, and thus the natural logarithm of $\mathrm{e}^x$ is $x$. We say the the functions $\mathrm{e}^x$ and $\ln x$ are inverse functions of each other: if we take $x$ and first apply $\mathrm{e}^x$, then $\ln x$, or the other way around, then the result is the same as if we did nothing to $x$ at all. In a sense, $\mathrm{e}^x$ and $\ln x$ cancel each other out.

Still considering only positive real number $x$, consider the power function $x^a$, where $a$ is any real number. We can use the fact that $\mathrm{e}^x$ and $\ln x$ are inverses of each other to rewrite the power function $x^a$:

$$x^a = \left(\mathrm{e}^{\ln x}\right)^a = \mathrm{e}^{a \ln x}.$$

This last expression, $\mathrm{e}^{a \ln x}$ is what we will use to define $x^a$ for positive real numbers $x$. That is, we define the quantity $x^a$ for positive real numbers $x$ and all real numbers $a$ to be e raised to the power of the quantity $a$ times the natural logarithm of $x$. In this way, we know how to compute, say, $3^\pi$, because we know what it means to take the natural logarithm of 3, multiply it by $\pi$, and then raise e to the power of that number. Thus, to the nearest thousandth, we have that

$$3^\pi = \mathrm{e}^{\pi \ln 3} = \mathrm{e}^{3.142 \cdot 1.099} = \mathrm{e}^{3.451} = 31.544.$$

You may be thinking at this point that we merely shifted the problem from understanding how to raise $x$ to the power of $a$ to understanding how to raise e to some power. How do we know how to calculate $\mathrm{e}^a$, where $a$ is any real number? There is an answer to this question, which is somewhat beyond the scope of this course, but here it is anyway: there is another way to write $\mathrm{e}^a$. Specifically, we can write $\mathrm{e}^a$ as

$$\mathrm{e}^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{6} + \frac{a^4}{24} + \cdots + \frac{a^n}{n!} + \cdots .$$

We will not tell you why $\mathrm{e}^a$ is equal to this infinite sum, this series as mathematicians would say, but it is true, and you can verify it using any calculator. Specifically, you can show that

$$\mathrm{e} = \mathrm{e}^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!} + \cdots ,$$

and, in fact, this is where the mysterious number e comes from. You do need to know all of this for this course, but it does justify why we know how to raise e to any power $a$: we can compute the infinite sum above (or, at least, approximate it) and get the same number.

So, at this point, we know how to define $x^a$ when $x$ is a positive real number. What about when $x$ equals 0? To define $0^a$, we take the right limit of $x^a$ as $x$ approaches 0. There are three cases:

- If $a > 0$, then $a \ln x$ approaches negative infinity as $x$ approaches 0 from the right. Therefore

$$\lim_{x \to 0^+} x^a = \lim_{x \to 0^+} \mathrm{e}^{a \ln x} = 0,$$

  since $\mathrm{e}^x$ has a left horizontal asymptote at $y = 0$. Thus we say that $0^a = 0$ when $a > 0$.

- If $a = 0$, then $x^0$ equals 1 for all positive values of $x$. Therefore the limit as $x$ approaches 0 of $x^0$ is 1, so we define $0^0 = 1$.

- If $a < 0$, then $a \ln x$ approaches positive infinity as $x$ approaches 0 from the right. Therefore

$$\lim_{x \to 0^+} x^a = \lim_{x \to 0^+} \mathrm{e}^{a \ln x} = +\infty,$$

  which means that the limit, for our purposes, is undefined. Therefore $0^a$ is undefined for $a < 0$.

So, to summarize:

$$0^a = \begin{cases} 0 & a > 0 \\ 1 & a = 0 \\ \text{undefined} & a < 0 \end{cases}.$$

Now for the difficult part: how do we define $x^a$ when $x$ is a negative number? The answer depends very strongly on the value of $a$, for reasons well beyond the scope of this class having to do with complex numbers. We state the definitions of $(-x)^a$ for $x > 0$ in terms of $x^a$, which we already know how to define, below:

- if $a$ is an even integer, then $(-x)^a = x^a$, so $x^a$ is an even function (as we already know).
- if $a$ is an odd integer, then $(-x)^a = -(x^a)$, making $x^a$ an odd function.
- if $a$ is a rational number, and, when we write $a$ as a reduced fraction, the denominator of $a$ is even, then $(-x)^a$ is undefined.
- if $a$ is a rational number, and, when we write $a$ as a reduced fraction, the denominator of $a$ is odd, then there are two possibilities:
  - if the numerator of the reduced fraction is even, then $(-x)^a = x^a$.
  - if the numerator of the reduced fraction is odd, then $(-x)^a = -(x^a)$.
- if $a$ is an irrational number, then $x^a$ is undefined.

So, for example:

- $(-2)^{-4} = 2^{-4} = \frac{1}{16}$.
- $(-2)^{-3} = -(2^{-3}) = -\frac{1}{8}$.
- to find $(-2)^{\frac{10}{16}}$, we write $\frac{10}{16}$ as a reduced fraction:

$$\frac{10}{16} = \frac{5}{8}.$$

  The denominator of $\frac{5}{8}$ is even, so $(-2)^{\frac{10}{16}}$ is undefined.

- to find $(-2)^{\frac{8}{18}}$, we write $\frac{8}{18}$ as a reduced fraction:

$$\frac{8}{18} = \frac{4}{9}.$$

  The denominator of $\frac{4}{9}$ is odd, and its numerator is even, so $(-2)^{\frac{8}{18}} = \left(2^{\frac{8}{18}}\right)$, whatever that number might be.

- to find $(-2)^{-4.6}$, we write $-4.6$ as a reduced fraction:

$$-4.6 = -\frac{46}{10} = -\frac{23}{5}.$$

  The denominator of $-\frac{23}{5}$ is odd, and so is the numerator, so $(-2)^{-4.6} = -(2^{4.6})$.

- $(-2)^\pi$ is undefined, since $\pi$ is an irrational number.

Now that we have fully defined the power functions, we will learn how to differentiate them.

## Differentiating General Power Functions

Let $f(x) = x^a$, where $a$ is any real number. Then, where $f(x)$ has a derivative, we have the formula:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = ax^{a-1}.$$

The difficulty here is not in the formula, with which we should be familiar by now. The question of interest is: at what points is the derivative of a power function defined?

Suppose $p$ is a value of $x$ in the domain of $x^a$, as determined in the previous section. If $p \neq 0$, then automatically $x^a$ has a derivative at $p$, and the formula above for the derivative applies. If $p = 0$, then the situation is a bit more complicated:

- if $a < 0$, then $x^a$ does not have a derivative at 0, since $x^a$ is not defined at 0.
- if $a = 0$, then $x^a$ has a derivative at 0.
- if $0 < a < 1$, then $x^a$ does not have a derivative at 0.
- if $a \geq 1$ and $x^a$ is defined for negative values of $x$, then $x^a$ has a derivative at 0.
- if $a \geq 1$ and $x^a$ is not defined for negative values of $x$, then $x^a$ does not have a derivative at 0.

Let us do a few examples to make this all clear. Consider the function $f(x) = x^\pi$. First, let us find the formula for the derivative of $f(x)$:

$$f'(x) = \pi x^{\pi-1}.$$

Next, we determine the domain of $f(x)$ and the points at which the derivative is defined. The number $\pi$ is positive, so $f(x)$ is defined at $x = 0$, and it is irrational, so $f(x)$ is not defined for negative numbers. Therefore we know that $f(x)$ has a derivative for all positive numbers, but not for 0.

Next, let us do a Chain Rule problem: let $g(x) = (\mathrm{e}^x - 1)^{-\frac{4}{7}}$. We first find the formula for the derivative at the points at which it exists:

$$g'(x) = -\frac{4}{7}(\mathrm{e}^x - 1)^{-\frac{4}{7}-1} \cdot \mathrm{e}^x = -\frac{4}{7}\mathrm{e}^x(\mathrm{e}^x - 1)^{-\frac{4}{7}-\frac{7}{7}} = -\frac{4}{7}\mathrm{e}^x(\mathrm{e}^x - 1)^{-\frac{11}{7}}.$$

Next we need to find the domain of $g(x)$. First, we find the domain of the outside function, which is $x^{-\frac{4}{7}}$. This function is defined automatically for all positive numbers; it is not defined for 0, since $-\frac{4}{7}$ is negative; and it is defined for negative values of $x$, since $-\frac{4}{7}$, which is already in reduced form, has an odd denominator. Therefore $g(x)$ is defined whenever the inside function, $\mathrm{e}^x - 1$, is not equal to 0. In other words, the condition is $\mathrm{e}^x \neq 1$, so $x \neq 0$. Thus the domain of $g(x)$ is

$$\mathrm{Dom}(g) = \{x \in \mathbb{R} : x \neq 0\}.$$

Finally, we could find where this composition has a derivative, but that is very complicated, so we will leave the question of differentiability alone.

For another example, take $h(x) = \sqrt{1 + x^2}$, which can also be written as $f(x) = (1 + x^2)^{\frac{1}{2}}$. First, we find a formula for its derivative:

$$f'(x) = \frac{1}{2}(1 + x^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{(1 + x^2)^{\frac{1}{2}}} = \frac{x}{\sqrt{1 + x^2}}.$$

As for the domain of $h(x)$, we note that $\sqrt{x}$ is only defined for $x \geq 0$, but $1 + x^2$ is always greater than or equal to 1, so $h(x)$ is defined everywhere. It also turns out that $h(x)$ has a derivative everywhere, but this is harder to show.

Finally, take $k(x) = \sqrt[6]{\cos x - 3}$, which we can also write as $(\cos x - 3)^{\frac{1}{6}}$. Sometimes it is better to find the domain of a function first, and this is one of those occasions. The outside function, $x^{\frac{1}{6}}$, is defined for $x \geq 0$, so in order for $x$ to be in the domain of $k(x)$, we must have that $\cos x - 3 \geq 0$. We can rewrite this as $\cos x \geq 3$, which is impossible, because the maximum value of $\cos x$ is 1. Therefore $k(x)$ is not defined anywhere, and so it is not differentiable anywhere either. This example illustrates the idea that being able to write down a formula does not automatically imply that that formula defines a function with a non-empty domain.