Logarithmic Functions

Logarithmic Functions and Their Properties

We now shift our attention back to classes of functions and their derivatives. Today we study logarithmic functions. A logarithmic function is a function of the form

$$f(x) = \log_a x,$$

where a is a positive real number not equal to 1. The logarithmic function $\log_a x$ takes an element of the domain x and gives back the unique number $b = \log_a x$ such that $a^b = x$. Notice that logarithmic functions are only defined for positive real numbers x, so the domain of a logarithmic function is

$$\mathrm{Dom}(\log_a x) = \{x \in \mathbb{R} \,:\, x > 0\}.$$

The most important logarithmic function is the natural logarithmic function

$$f(x) = \ln x.$$

Let us graph the natural logarithmic function using the numerical table below (with values given to the nearest hundredth:

x	$\ln x$
0.25	-1.39
0.50	-0.70
1.00	0.00
2.00	0.70
4.00	1.39

The graph that we get has several important properties. First, since the domain of $\ln x$ is all positive real numbers, the graph lies entirely to the right of the y-axis. Second, were we to plot points closer and closer to the y-axis, we see that the graph of the natural logarithmic function has a right negative vertical asymptote, that is

$$\lim_{x \to 0^+} \ln x = -\infty.$$

Third, the graph of the natural logarithmic function crosses the x-axis at the point $(1, 0)$, which makes sense, because the natural logarithm (or any logarithm) of a number is 0, then that number equals e^0 (or a^0), which we know is 1. Fourth, the graph of the natural logarithmic function is an increasing function everywhere, and, were we to plot the natural logarithm of larger and larger positive numbers, we would see that the function would got to positive infinity, but very slowly, much slower than any other function we have seen in this class. This makes sense: e^{10} is about 22026, and yet it only has a natural log of 10. Natural log can get as big as you want it to be, but you have to input very very large numbers in order to get large numbers as your output.

In general, the logarithmic functions $\log_a x$ have the following properties:

- **Positive Domain:** As stated before, all logarithmic functions are defined only for positive numbers. For completeness, we state this again in set notation:

$$\mathrm{Dom}(\log_a x) = \{x \in \mathbb{R} \,:\, x > 0\}.$$

- **Horizontal Intercept:** For all positive numbers a, we know that $a^0 = 1$. Therefore for all logarithmic functions $\log_a x$, the only solution to the equation $\log_a x = 0$ is $x = 1$. Thus we have that all logarithmic functions intersect have a horizontal intercept (that is, have an x-intercept) of 1. So the graphs of all logarithmic function intersect at $(1, 0)$.

- **Right Vertical Asymptote:** All logarithmic functions have a right vertical asymptote at $x = 0$. Whether that right vertical asymptote is positive or negative depends on whether $a > 1$ or $a < 1$. If $a > 1$ then, as we saw with the natural logarithm, for which $a = e > 1$, we have that

$$\lim_{x \to 0^+} \log_a x = -\infty.$$

  For $a < 1$, we have the opposite:

$$\lim_{x \to 0^+} \log_a x = +\infty.$$

  Sketch out the graph of $\log_{0.5} x$ to confirm this.

- **Slow Growth:** When $a > 1$, the function $\log_a x$ will get larger and larger, approaching positive infinity, but it will do so very slowly. Likewise, when $a < 1$, the function $\log_a x$ will get more and more negative, approaching negative infinity, but it too will do so very slowly.

Differentiating Logarithmic Functions

If you look at the graph of $\ln x$, you will notice the slope of the graph is always positive, and gets closer and closer to 0 as x gets more and more positive. Thus we can expect that the derivative of the natural logarithm function will have a right horizontal asymptote to the line $y = 0$, and indeed this is the case.

Let $f(x) = \log_a x$, where a is a positive number not equal to 1. The derivative of $f(x)$ is then

$$\frac{df}{dx} = \frac{1}{(\ln a)x}.$$

In particular, if we take $a = e$, then the derivative of the natural logarithmic function is given by

$$\frac{d}{dx}(\ln x) = \frac{1}{(\ln e)x} = \frac{1}{x}.$$

So, apparently, logarithmic functions and negative power functions are somehow related to each other. This formula explains why, as x gets more and more positive, the slope of the graph of the natural logarithmic function approaches 0.

There is one point we need to make about this formula for the derivative of logarithmic functions, and that is that this formula is only defined where the logarithmic functions are defined, which is for positive real numbers x. So, when drawing the graph of the derivative of $\ln x$, you only draw the right half of the graph of x^{-1}. If you draw the part of the graph of x^{-1} which is to the left of the y-axis, then you are saying that $\ln x$ has a derivative for negative values of x, and since $\ln x$ is not defined for negative values of x, you would be wrong. Be careful!

Let us do some examples. Let $f(x) = \log_{10} x$. Then the formula above tells us that

$$f'(x) = \frac{1}{(\ln 10)x}.$$

If we took $g(x) = \log_{0.5} x$, then we would have that

$$g'(x) = \frac{1}{(ln 0.5)x}.$$

The natural log of 0.5 is a negative number, so this tells us that the derivative of $g(x)$ is always negative. Based on your sketch of $\log_{0.5} x$, is this what you expected?

Now let us do a couple of examples of using the Chain Rule with logarithmic functions. Let $h(x) = \ln(x^2)$. First, let us examine the domain of this composition. The outside function is $\ln x$, and we know that to be in the domain of $\ln x$, x must be a positive number. This tells us that the only x which can be in the domain of $\ln(x^2)$ are those for which x^2 is a positive number. The function x^2 is positive as long as $x \neq 0$, so we get that

$$\text{Dom}(h) = \{x \in \mathbb{R} \: : \: x \neq 0\}.$$

Now we take the derivative using the Chain Rule:

$$h'(x) = \frac{1}{x^2} \cdot 2x = \frac{2x}{x^2}.$$

This formula for the derivative is defined everywhere on the domain of $h(x)$, and so $h(x)$ is differentiable everywhere. Incidentally, you can leave this formula for $h'(x)$ as it stands now, or you can note that, since $x \neq 0$, we can divide by x and simplify the expression to

$$h'(x) = \frac{2}{x}.$$

Whether you do so or not is your choice.

For our next example, take $k(x) = \ln(\sin x)$. Again, let us find the domain of this composition first, and then take its derivative. To find the domain of this composition, we start with the domain of the outside function, which, again, is $\ln x$. The domain of $\ln x$ is all positive real numbers, so this tells us that if x is in the domain of $k(x)$, then $\sin x$ must be positive. Where is $\sin x$ a positive function? Looking at our unit circle, we see that it is positive on the open interval $0 < x < \pi$, but also for $2\pi < x < 3\pi$, and $-2\pi < x < -\pi$, and an assortment of other open intervals. This is a difficult domain to write down: one way we can write it in set notation is

$$\text{Dom}(k) = \{x \in \mathbb{R} \ : \ n\pi < x < (n+1)\pi, \ \text{where } n \text{ is an even integer}\}.$$

Now let us use the Chain Rule to find the derivative of $k(x)$:

$$k'(x) = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x.$$

So the derivative of $k(x)$, where it is defined, is equal to the cotangent of x. The function $\cot x$ is defined everywhere except where $\sin x = 0$. There are no values of x in the domain of $k(x)$ for which $\sin x = 0$, and so $k(x)$ is differentiable everywhere it is defined. What does the graph of $k'(x)$ look like, and how does it compare to the graph of $\cot x$? Can you think of a way to get a function whose derivative is $\tan x$ everywhere it is defined?

Let us do one more example: take $j(x) = \ln|x|$. First we find its domain: the domain of the outside function $\ln x$ tells us that for x to be in the domain of $j(x)$, we must have that $|x| > 0$. This inequality is true for all x except 0, so the domain of $j(x)$ is

$$\text{Dom}(j) = \{x \in \mathbb{R} \ : \ x \neq 0\}.$$

Now we find the derivative of $j(x)$ using the Chain Rule:

$$j'(x) = \frac{1}{|x|} \cdot \frac{\text{d}}{\text{d}x}(|x|).$$

To finish off this formula, we need to use the case-by-case definition of the absolute value function:

$$|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \ . \\ -x & x < 0 \end{cases}$$

This tells us that the derivative of $|x|$ is 1 when $x > 0$ and -1 when $x < 0$ (since 0 is not in the domain of $j(x)$, we are unconcerned about $|x|$ not having a derivative at $x = 0$). So we break up our formula for $j'(x)$ into cases: when $x > 0$, we have that

$$j'(x) = \frac{1}{|x|} \cdot \frac{\text{d}}{\text{d}x}(|x|) = \frac{1}{x} \cdot 1 = \frac{1}{x},$$

and when $x < 0$, we get that

$$j'(x)\frac{1}{|x|} \cdot \frac{\text{d}}{\text{d}x}(|x|) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}.$$

So, for all x in the domain of $j(x)$, we have that

$$j'(x) = \frac{1}{x}.$$

So, while the derivative of $\ln x$ is x^{-1} restricted to the positive real numbers, the derivative of $\ln |x|$ is x^{-1} everywhere that it is defined. You will use this derivative for $\ln |x|$ very often when you study integration next term.