Differentiability

The Difference Quotient and the Derivative

Today we will study what it exactly means for a function to have a derivative. In doing so, we will talk about a specific rational function, called the difference quotient, and how it it used to define the derivative. We will discuss functions which are continuous at a point but do not have a derivative at that point, and we will talk about functions defined on a closed interval and how being defined on a closed interval impacts the search for maxima and minima.

Let $f(x)$ be any real valued function, and let a be a value of x at which $f(x)$ is defined. We will define a new function $\delta(x)$ by

$$\delta(x) = \frac{f(x) - f(a)}{x - a}.$$

The function $\delta(x)$ is called the difference quotient of $f(x)$ at $x = a$. It is defined everywhere $f(x)$ is defined except for $x = a$:

$$\mathrm{Dom}(\delta) = \{x \in \mathbb{R} \,:\, x \in \mathrm{Dom}(f),\, x \neq a\}.$$

You may recognize this difference quotient as being very much like the slope formulae: indeed, we have seen a formula like this once before. Geometrically, we can represent $\delta(p)$ for some value p of x at which $\delta(x)$ is defined as being the slope of the line passing through the graph of $f(x)$ at $x = p$ and at $x = a$. If you are reading these notes on your own, you should sketch out the graph of a typical function $f(x)$, pick a value for a and a value of p, and draw the line passing through the points $(a, f(a)$ and $(p, f(p))$. The slope of this line then represents $\delta(p)$.

Given that our last lecture dealt with rational functions and how to find their limits at points outside of their domains, it should seem natural to you that we are going to find the limit of $\delta(x)$ as x approaches a. Geometrically, we see this limit as being the slope of the line we drew above as p gets closer and closer to a from either side. As p gets closer and closer, that line begins to look more and more like the tangent line to $f(x)$ at $x = a$. Since $\delta(x)$ represents the slope of this line, the limit of $\delta(x)$ as x approaches a from either side, if it exists, will equal the slope of this tangent line. We have a name for the slope of the tangent line to $f(x)$ at $x = a$: the derivative. Therefore we write:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This is the definition of the derivative of $f(x)$ at $x = a$. It is the limit of a rational function, the difference quotient of $f(x)$ at $x = a$. We say that $f(x)$ is differentiable at $x = a$ if this limit exists. If this limit does not exist, we say that a is a point of non-differentiability for $f(x)$. If $f(x)$ is differentiable at every point in its domain, we say that $f(x)$ is a differentiable function on its domain. So far, every function we have studied in this class, with the exception of the piecewise continuous functions, is differentiable everywhere on its domain: this includes polynomial functions, the trigonometric functions (we emphasize that we are talking about having a derivative where they are defined), the exponential functions, the power functions, and all stretches and shifts of these functions.

Now that we have a limit definition for the derivative and a reasonable idea of how to take a limit, we are going to discuss the derivatives of some basic functions again in this context. All of the formulae you have learned in this class for the derivatives of functions can be derived from the limit definition of the derivative. For example, consider a constant function $f(x) = c$. Plugging this formula into the limit definition for the derivative, we get that

$$\frac{\mathrm{d}f}{\mathrm{d}x}(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{c - c}{x - a} = \lim_{x \to a} \frac{0}{x - a}.$$

Now, the function $\frac{0}{x-a}$ is the zero function everywhere except at $x = a$, where it is undefined. Therefore that last limit is equal to 0. Thus we have shown that

$$\frac{\mathrm{d}f}{\mathrm{d}x}(a) = 0,$$

which is precisely the formula that we have been using throughout this course.

Let us try another simple example: take $f(x) = mx+b$. Substituting this formula into the limit definition, we get

$$\frac{\mathrm{d}f}{\mathrm{d}x}(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{(mx+b)-(ma+b)}{x-a} = \lim_{x\to a} \frac{mx-ma}{x-a} = \lim_{x\to a} \frac{m(x-a)}{x-a}.$$

This rightmost function is a rational polynomial function which equals m everywhere on its domain, which is all real numbers except a. Thus this last limit is equal to m, so that

$$\frac{\mathrm{d}f}{\mathrm{d}x}(a) = m.$$

Finally, let us consider the function $f(x) = \mathrm{e}^x$. We will find its derivative at a specific point, $a = 1$, with a numerical table. First, let us write out the limit we need to find:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(1) = \lim_{x\to 1} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\mathrm{e}^x-\mathrm{e}^1}{x-1} = \lim_{x\to 1} \frac{\mathrm{e}^x-\mathrm{e}}{x-1} = \lim_{x\to 1} \delta(x).$$

Now let us make a table to estimate this limit. The entries in the table are accurate to the nearest millionth:

x	$\delta(x)$
0.000000	1.718282
0.900000	2.586787
0.990000	2.704736
0.999000	2.716923
0.999900	2.718146
0.999990	2.718268
0.999999	2.718280

x	$\delta(x)$
2.000000	4.670774
1.100000	2.858842
1.010000	2.731919
1.001000	2.719641
1.000100	2.718418
1.000010	2.718295
1.000001	2.718283

Considering that the number is e is 2.718282 to the nearest millionth, it seems reasonably clear that this limit exists and equals e. This is consistent with our formula for the derivative of e^x, which tells that

$$\frac{\mathrm{d}f}{\mathrm{d}x}(1) = \mathrm{e}^1 = \mathrm{e}.$$

You may be wondering why we do not use L'Hôpital's Rule to find the limit above. Technically, we could use L'Hôpital's Rule, but then we would be using the formula for the derivative of e^x to find the derivative of e^x. L'Hôpital's Rule relies on the limit definition of the derivative, so we cannot use it to find the derivative of a function, or to show that a function has a derivative.

Points of Non-Differentiability

When is a function not differentiable at a point? The first answer to this question is that a function is not differentiable at a point if it is not continuous at that point. Consider again the limit definition of the derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

The denominator of the difference quotient goes to 0 as x approaches a, so in order for this limit to exist, the numerator must also go to 0 as x approaches a, that is

$$\lim_{x\to a} (f(x) - f(a)) = 0.$$

Putting the constant term $f(a)$ on the right hand side of the equation, we get that

$$\lim_{x\to a} f(x) = f(a),$$

which is our way of saying in limit notation that $f(x)$ is continuous at $x = a$. So if the derivative of $f(x)$ exists at $x = a$, $f(x)$ must be continuous at $x = a$ as well. Thus, if $f(x)$ is discontinuous at $x = a$, it is also non-differentiable at $x = a$.

If $f(x)$ is continuous at $x = a$, it does not follow that $f(x)$ is differentiable at $x = a$. The most famous example of this is the absolute value function:

$$f(x) = |x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}.$$

The graph of the absolute value function looks like the line $y = x$ for positive x and $y = -x$ for negative x. Both of these functions have a y-intercept of 0, and since the function is defined to be 0 at $x = 0$, the absolute value function is continuous. That said, the function $f(x) = |x|$ is not differentiable at $x = 0$. Consider the limit definition of the derivative at $x = 0$ of the absolute value function:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}.$$

If this limit exists, then the left limit must equal the right limit. The left limit is given by

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1,$$

while the right limit is given by

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{x}{x} = 1.$$

The left limit does not equal the right limit, and therefore the limit of the difference quotient of $f(x) = |x|$ at $x = 0$ does not exist. Thus the absolute value function is not differentiable at $x = 0$. We can also see this by looking at the graph of $|x|$: at $x = 0$, the graph of $f(x) = |x|$ has a "corner" in it, a place where the direction of the curve abruptly changes. In general, this is the hallmark of a point of non-differentiability on a continuous function: if the graph of $f(x)$ has a corner at $x = a$, then $f(x)$ is not differentiable at $x = a$.

Another way in which a function can have non-differentiable points is the following: take your favorite continuous function $f(x)$, and restrict its domain to some closed interval $[c, d]$. By restricting the domain of a function, we mean that we are only going to allow the real numbers from c to d, including c and d to be inputs to our function. So, for example, take the absolute value function $f(x) = |x|$ and restrict it to the closed interval $[-1, 2]$. We already know that this function, with this new domain, has at least one point of non-differentiability: $f(x)$ is not differentiable at $x = 0$. It turns out that it is also not differentiable at the endpoints of the closed interval, that is, at $x = -1$ and $x = 2$. Why is this? The derivative of $f(x)$ is a limit, which, again, means that if it exists, then the left limit and the right limit must also exist and must be equal to each other. For $x = -1$, the right limit exists, but the restricted function is not defined to the left of -1, and so the left limit cannot exist, and thus the derivative also does not exist. Likewise, at $x = 2$, the left limit of the difference quotient exists, but the function is undefined to the right of $x = 2$, so the right limit does not exist. So $f(x) = |x|$ restricted to $[-1, 2]$ is not differentiable at $x = -1$, $x = 0$, and $x = 2$. In general, if $f(x)$ is restricted to the closed interval $[c, d]$, then automatically $x = c$ and $x = d$ will be points of non-differentiability.

Finally, let us discuss maxima and minima and points of non-differentiability. Consider again $f(x) = |x|$ restricted to the closed interval $[-1, 2]$. What is the maximum value of $|x|$ on this interval? It is 2, and it occurs at $x = 2$. What is its minimum value on this interval? That would be 0, and it occurs at $x = 0$. What is interesting about both the maximum value and the minimum value of $|x|$ on $[-1, 2]$ is that neither occur at critical points, places where the derivative is 0. In fact, $f(x) = |x|$ does not have any critical points.

We used to say that if $f(x)$ has a local maximum or a local minimum at a, then $x = a$ must be a critical point for $f(x)$. This is true when we are dealing with functions which are differentiable everywhere on their domains; it is no longer true when $f(x)$ has points of non-differentiability. We now say that if $f(x)$ has a local maximum or a local minimum at a, then $x = a$ must be either be a critical point or a point of non-differentiability. So, in order to find the local maxima and local minima of a function, you have to check its critical points and the points at which its derivative is undefined.

One last point: suppose we restrict a continuous function $f(x)$ to a closed interval $[c, d]$. It is a theorem, a proven fact, that $f(x)$ must have both a maximum value and a minimum value on the closed interval $[c, d]$

(among the values of the function on that interval). For example, if we take $f(x) = x^2$, we know that this function does not have a maximum value on the real line, but if we restrict to the closed interval $[-2, 3]$, then it has a maximum value. We find this maximum value by finding all of the critical points of x^2 on this closed interval (so, $x = 0$), and all of the points of non-differentiability (in this case, only the endpoints $x = -2$ and $x = 3$). We then find the values of the function at these three points and compare them:

x	$f(x)$
-2	4
0	0
3	9

The maximum of these three values of the function is 9, so x^2 restricted to $[-2, 3]$ achieves its maximum at $x = 3$; it achieves its minimum at $x = 0$, at the critical point. What would have happened to the minimum of the function had $f(x) = x^2$ been restricted to $[1, 3]$?