Rational Functions

Rational Functions and Their Domains

Our last lecture was devoted to rational polynomial functions, which, if you recall, are the functions which are the quotient of two polynomials. Today we discuss rational function in general. A function $h(x)$ is a rational function if it is the quotient of two other functions $f(x)$ and $g(x)$:

$$h(x) = \frac{f(x)}{g(x)}.$$

For the remainder of this lecture we will assume that $f(x)$ and $g(x)$ are real valued functions with derivatives everywhere that they are defined. If we further assume that $f(x)$ and $g(x)$ are defined for all real numbers, then the domain of $h(x)$ is equal to all real numbers x such that $g(x) \neq 0$:

$$\text{Dom}(h) = \{x \in \mathbb{R} \, : \, g(x) \neq 0\}.$$

So, for example, the function

$$h(x) = \frac{\sin x}{x}$$

is defined for all real numbers not equal to 0. We also know how to find the derivative of a rational function wherever it is defined: we use the quotient rule. In our example above, we find that the derivative of $h(x)$, where it is defined, is

$$h'(x) = \frac{x \cdot \cos x - \sin x \cdot 1}{x^2} = \frac{x \cos x - \sin x}{x^2}.$$

All of the above concepts are review for us. Today we will discuss how to find the left and right limits of a rational function, when they exist.

Limits of Rational Functions and L'Hôpital's Rule

Let $h(x)$ be a rational function, the quotient of $f(x)$ and $g(x)$ as defined above. Let a be a value of x inside the domain of $h(x)$. Then, assuming that $f(x)$ and $g(x)$ are continuous functions at a, $h(x)$ will be a continuous function at a as well, and so $h(x)$ will have a limit at $x = a$, and that limit will equal $h(a)$:

$$\lim_{x \to a} h(x) = h(a) = \frac{f(a)}{g(a)}.$$

For example, we know that $\frac{\sin x}{x}$ is continuous at $x = \pi$, since both $\sin x$ and x are continuous functions, so the limit of $\frac{\sin x}{x}$ at $x = \pi$ is

$$\lim_{x \to \pi} \frac{\sin x}{x} = \frac{\sin \pi}{\pi} = \frac{0}{\pi} = 0.$$

Let us assume for the rest of this section that $f(x)$ and $g(x)$ are continuous everywhere and have derivatives everywhere. The only values of x such that $h(x)$ is not defined are those values for which $g(x) = 0$. Let a be such a value. Then $h(x)$ may or may not have left and right limits at $x = a$. One possibility that we already discussed is that $h(x)$ could have left and right vertical asymptotes at a. An example of this phenomenon is $h(x) = \frac{x^2-6x+5}{x+1}$, which we showed during the last lecture to have both a left negative vertical asymptote and a right positive vertical asymptote at $x = -1$. The reason why this particular rational polynomial function has vertical asymptotes as opposed to limits is that, while its denominator approaches 0 as x approaches -1, its numerator approaches some other number, so that as the x approaches -1 from either side, $h(x)$ is the quotient of some relatively large number (by relatively large number we mean a number which we know is not close to 0) and some very small number, so that $h(x)$ either approaches negative infinity or positive infinity, depending on the direction from which x is approaching. In general, if $h(x) = \frac{f(x)}{g(x)}$ and a is a point for which $g(a) = 0$, then if $f(a) \neq 0$ then $h(x)$ will have vertical asymptotes at $x = a$. If $f(a) = 0$, however, then that is another story.

Consider $h(x) = \frac{\sin x}{x}$. The function $h(x)$ is not defined at 0. We also have that the numerator, $\sin x$, is equal to 0 at $x = 0$. So, at first glance, it looks like it is going to be difficult to find the limit of $h(x)$ at $x = 0$, if it even exists. One way we could do this is to find the limit numerically, that is, find $h(x)$ for values closer and closer to 0 and make a guess about the limit. We do this below from both directions for $\frac{\sin x}{x}$ (the values below are rounded to the nearest millionth):

x	$\sin x$	$h(x)$
1.000000	0.841471	0.841471
0.100000	0.099833	0.998334
0.010000	0.009998	0.999833
0.001000	0.001000	1.000000
0.000100	0.000100	1.000000

x	$\sin x$	$h(x)$
−1.000000	−0.841471	0.841471
−0.100000	−0.099833	0.998334
−0.010000	−0.009998	0.999833
−0.001000	−0.001000	1.000000
−0.000100	−0.000100	1.000000

It should be clear to you from these numerical tables that the limit of $\frac{\sin x}{x}$ as x approaches 0 from either side is 1.

Now, for many functions, this method of computing the values of the function for values of x closer and closer to a and then guessing the limit is the only way to get a reasonable estimate of that limit. The problem with this method is that it is only a guess, and sometimes guesses are wrong: who is to say that, had we chosen values of x even closer to 0 in the tables above that $h(x)$ would have suddenly diverged from 1? This can and will happen from time to time, so if we have a method which can tell us the limit of $h(x)$ as x approaches a exactly, then that method is a very powerful tool for us. We do have a tool like this: we call it L'Hôpital's Rule.

The statement of L'Hôpital's Rule is the following: suppose

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x)$ and $g(x)$ are continuous functions with derivatives everywhere. Let a be a value of x such that $f(a) = 0$ and $g(a) = 0$ (which means that $h(a)$ is not defined). Then, if the limit (or left limit, or right limit) of $h(x)$ as x approaches a exists, it is equal to the limit (or left limit, or right limit) of the quotient of $f'(x)$ and $g'(x)$, if it exists:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Let us use $h(x) = \frac{\sin x}{x}$ as an example. The derivative of the numerator of $h(x)$ is $\cos x$, and the derivative of the denominator is 1. L'Hôpital's Rule tells us that the limit as x goes to 0 of $h(x)$ is equal to the limit as x goes to 0 of the quotient of the derivative of the numerator and the derivative of the denominator, which is $\frac{\cos x}{1}$, or simply $\cos x$. Thus we have by L'Hôpital's Rule that

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1,$$

since $\cos x$ is a continuous function. This is the value that we guessed the limit would be.

Let us try another example: let

$$h(x) = \frac{\mathrm{e}^x - 1}{6x},$$

and let us try to find the limit of $h(x)$ at $x = 0$ again. Just as before, we have that both the numerator and the denominator are equal to 0 at $x = 0$, and that both the numerator and the denominator are continuous functions with derivatives everywhere. Therefore we can apply L'Hôpital's Rule: if the quotient of the derivatives of the numerator and the denominator has a limit at $x = 0$, then that limit is equal to the limit of $h(x)$ at $x = 0$:

$$\lim_{x \to 0} \frac{\mathrm{e}^x - 1}{6x} = \lim_{x \to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x - 1)}{\frac{\mathrm{d}}{\mathrm{d}x}(6x)} = \lim_{x \to 0} \frac{\mathrm{e}^x}{6} = \frac{\mathrm{e}^0}{6} = \frac{1}{6}.$$

Since the quotient of the derivatives of the numerator and the denominator has a limit of $\frac{1}{6}$ at $x = 0$, so does $h(x)$.

Sometimes we have to make use of L'Hôpital's Rule more than once in a problem to find a limit. Consider

$$h(x) = \frac{x^3 + x^2 - 8x - 12}{x^2 + 4x + 4}.$$

Let us find the limit of $h(x)$ at $x = -2$, if it exists. This is a rational polynomial functions, and so both the numerator and the denominator are continuous functions with derivatives everywhere. we also see, by substituting -2 in for x that both the numerator and the denominator are 0 at $x = -2$. Applying L'Hôpital's Rule, we get that

$$\lim_{x \to -2} \frac{x^3 + x^2 - 8x - 12}{x^2 + 4x + 4} = \lim_{x \to -2} \frac{3x^2 + 2x - 8}{2x + 4}.$$

Here we have a problem: both the derivative of the numerator and the derivative of the denominator are 0 at $x = -2$, so the limit of the quotient of the derivatives tells us nothing new. We do, however, have one thing going for us, and that is that both the derivative of the numerator and the derivative of the denominator are continuous functions with derivatives everywhere, so we can apply L'Hôpital's Rule again:

$$\lim_{x \to -2} \frac{x^3 + x^2 - 8x - 12}{x^2 + 4x + 4} = \lim_{x \to -2} \frac{3x^2 + 2x - 8}{2x + 4} = \lim_{x \to -2} \frac{6x + 2}{2} = \frac{6 \cdot (-2) + 2}{2} = \frac{-10}{2} = -5.$$

On our second application of L'Hôpital's Rule, we get a quotient which is a continuous function, and thus we find the limit. In practice, you may apply L'Hôpital's Rule as many times as necessary, as long as you apply it correctly each time.

Finally, we would like to point out that sometimes L'Hôpital's Rule can give us evidence of a quotient having a left or right vertical asymptote, even if both the numerator and the denominator are 0 at the point at which we are taking the limit. Consider

$$h(x) = \frac{x}{1 - \cos x}.$$

Let us find the limit, if it exists, of $h(x)$ at $x = 0$. This rational function meets all of the criteria for applying L'Hôpital's Rule, but when we apply it we get that

$$\lim_{x \to 0} \frac{x}{1 - \cos x} = \lim_{x \to 0} \frac{1}{\sin x},$$

and the limit on the right side of the equation does not exist, since the derivative of the numerator is 1 everywhere, while the derivative of the denominator is 0 at $x = 0$. Technically, L'Hôpital's Rule does not tell us anything in this case, because L'Hôpital's Rule is predicated on the existence of the limit of the right hand side of this equation. That said, this is strong evidence that $h(x)$ has vertical asymptotes, and, in fact, in does:

$$\lim_{x \to 0^-} h(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 0^+} h(x) = +\infty.$$

How would you go about convincing yourself that these vertical asymptotes exist?