Tangent, Cotangent, Secant, and Cosecant

The Quotient Rule

In our last lecture, among other things, we discussed the function $\frac{1}{x}$, its domain and its derivative. We also showed how to use the Chain Rule to find the domain and derivative of a function of the form

$$k(x) = \frac{1}{g(x)},$$

where $g(x)$ is some function with a derivative. Today we go one step further: we discuss the domain and the derivative of functions of the form

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x)$ and $g(x)$ are functions with derivatives. The rule of differentiation we will derive is called the quotient rule. We will then define the remaining trigonometric functions, and we will use the quotient rule to find formulae for their derivatives.

The quotient rule has the following statement: let $f(x)$ and $g(x)$ be two functions with derivatives. Then we can define a function

$$h(x) = \frac{f(x)}{g(x)}$$

which has domain

$$\mathrm{Dom}(h) = \{x \in \mathbb{R} : g(x) \neq 0\}$$

and which is differentiable everywhere on its domain, with the formula

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

We usually call $f(x)$ the top function, and $g(x)$ the bottom function, and so the formula for the derivative of the quotient $h(x)$ is given by "bottom times the derivative of the top minus top times the derivative of the bottom all over bottom squared." Try to remember this phrase, or one like it.

Let us study the quotient rule. First, we can rewrite the function $h(x)$ as

$$h(x) = \frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}.$$

Thus, we can think of the quotient $h(x)$ as being a product, with the first function being $f(x)$ and the second the composition of x^{-1} after $g(x)$.

To find the domain of $h(x)$, we first note that since $f(x)$ presumably has full domain (all real numbers), the domain of $h(x)$ will be precisely the same as that of $(g(x))^{-1}$ (since, wherever $(g(x))^{-1}$ is defined, $f(x)$ will also be defined, and so their product will be defined there as well). We find the domain of $(g(x))^{-1}$, a composition, by first looking at the domain of its outside function, x^{-1}: the domain of x^{-1} is all real numbers except 0. This tells us that, presuming $g(x)$ is defined everywhere, $(g(x))^{-1}$ is defined at all real numbers except those x for which $g(x) = 0$. Thus the domain of $h(x)$, which is the domain of $(g(x))^{-1}$, is the set of all real numbers x such that $g(x) \neq 0$, which is precisely what the quotient rule tells us.

Now, we can see that $h(x)$ can be written as a product. This means that, wherever $h(x)$ is defined, we can apply the product rule to find its derivative:

$$h'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \cdot \frac{1}{g(x)}\right) = f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{g(x)}\right).$$

To find the derivative of $(g(x))^{-1}$, we apply the Chain Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{g(x)}\right) = -(g(x))^{-2} \cdot g'(x) = -\frac{g'(x)}{(g(x))^2}.$$

Combining the two fractions under a common denominator, we get the derivative formula for the quotient rule:

$$\begin{aligned} h'(x) &= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{g(x)}\right) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} \\ &= \frac{g(x)f'(x)}{(g(x))^2} - \frac{f(x)g'(x)}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}. \end{aligned}$$

Let us do an example of the quotient rule. Let

$$h(x) = \frac{x+3}{x^2 - 4x + 5}.$$

The domain of $h(x)$ is all real numbers x such that the denominator, $x^2 - 4x + 5$, is not 0. The denominator is 0 precisely when $x = 1$ or $x = 5$, so this gives us that

$$\mathrm{Dom}(h) = \{x \in \mathbb{R} : x \neq 1, x \neq 5\}.$$

For any x in the domain of h, the function $h(x)$ has a derivative given by the quotient rule. That derivative is

$$h'(x) = \frac{(x^2 - 4x + 5) \cdot 1 - (x+3) \cdot (2x - 4)}{(x^2 - 4x + 5)^2} = \frac{x^2 - 4x + 5 - 2x^2 - 2x + 12}{(x^2 - 4x + 5)^2} = \frac{-x^2 - 6x + 17}{(x^2 - 4x + 5)^2}.$$

The function $h(x)$ is an example of a rational polynomial function. We will be studying rational polynomial functions later in the course.

The Other Trigonometric Functions

So far in this course, the only trigonometric functions which we have studied are sine and cosine. Today we discuss the four other trigonometric functions: tangent, cotangent, secant, and cosecant. Each of these functions are derived in some way from sine and cosine. The tangent of x is defined to be its sine divided by its cosine:

$$\tan x = \frac{\sin x}{\cos x}.$$

The cotangent of x is defined to be the cosine of x divided by the sine of x:

$$\cot x = \frac{\cos x}{\sin x}.$$

The secant of x is 1 divided by the cosine of x:

$$\sec x = \frac{1}{\cos x},$$

and the cosecant of x is defined to be 1 divided by the sine of x:

$$\csc x = \frac{1}{\sin x}.$$

If you are not in lecture today, you should use these formulae to make a numerical table for each of these functions and sketch out their graphs. Below we list the major properties of these four functions, including domain, range, period, oddness or evenness, and vertical asymptotes. None of these functions have horizontal asymptotes. You should verify that your sketches reflect these properties:

- **Tangent:** The function $\tan x$ is defined for all real numbers x such that $\cos x \neq 0$, since tangent is the quotient of sine over cosine. Thus $\tan x$ is undefined for

$$x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

Its range is all real numbers, that is, for any number y, you can always find a number x such that $y = \tan x$. The period of $\tan x$ is π. This is a departure from $\sin x$ and $\cos x$, which have periods of 2π. The reason is simple: opposite angles on the unit circle (like $\frac{\pi}{4}$ and $\frac{5\pi}{4}$) have the same tangent because of the signs of their sines and cosines. For example:

$$\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{\sin\frac{5\pi}{4}}{\cos\frac{5\pi}{4}} = \tan\frac{5\pi}{4}.$$

The function $\tan x$ is an odd function, which you should be able to verify on your own. Finally, at the values of x at which $\tan x$ is undefined, $\tan x$ has both left and right vertical asymptotes. Specifically, if a is a value of x outside the domain of $\tan x$, then

$$\lim_{x\to a^-}\tan x = +\infty \qquad \text{and} \qquad \lim_{x\to a^+}\tan x = -\infty.$$

- **Cotangent:** The function $\cot x$ is a lot like $\tan x$. It is defined at all values of x for which $\sin x \neq 0$. In other words, the domain of $\cot x$ is all real numbers x except

$$x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$$

Just as in the case of $\tan x$, the range of $\cot x$ is all real numbers, and this should not be surprising, since essentially $\cot x$ is 1 divided by the tangent of x. For this same reason, the period of $\cot x$ is also π instead of 2π. You can also verify that it is an odd function. Finally, like $\tan x$, the function $\cot x$ has left and right vertical asymptotes at each point at which it is undefined. If a is a value of x at which $\cot x$ is undefined, then we have that

$$\lim_{x\to a^-}\tan x = -\infty \qquad \text{and} \qquad \lim_{x\to a^+}\tan x = +\infty.$$

- **Secant:** The function $\sec x$ is defined to be the multiplicative inverse of cosine, so it is defined precisely where $\cos x$ is not equal to 0. So the domain of $\sec x$ is all real numbers x except

$$x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

Thus, $\sec x$ and $\tan x$ have the same domains. The range of $\sec x$ is a bit more complicated: remember that the bounds on $cosx$ are $-1 \leq \cos x \leq 1$. So, we see that if the secant of x is positive, then it can be no smaller than 1, and if it is negative, it can be no larger than -1. Thus the range of $\sec x$ is made up of two intervals:

$$\sec x \geq 1 \qquad \text{or} \qquad \sec x \leq -1.$$

The period of $\sec x$ is precisely the same as that of $\cos x$, which means that the period of $\sec x$ is 2π. The function $\sec x$ is an even function, and this is because $\cos x$ is an even function. Finally, at every value of x not in the domain of $\sec x$, the function has both left and right vertical asymptotes. If $a = \ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$ then the left vertical asymptote at a goes to positive infinity and the right vertical asymptote goes to negative infinity:

$$\lim_{x\to a^-}\sec x = +\infty \qquad \text{and} \qquad \lim_{x\to a^+}\sec x = -\infty.$$

If, on the other hand, $b = \ldots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \ldots$, then the left vertical asymptote at b goes to negative infinity and the right vertical asymptote goes to positive infinity:

$$\lim_{x\to b^-}\sec x = -\infty \qquad \text{and} \qquad \lim_{x\to b^+}\sec x = +\infty.$$

- **Cosecant:** Similar to the case of $\sec x$, the function $\csc x$ is defined precisely when $\sin x$ is not equal to 0. Thus the values of x at which $\csc x$ is undefined are

$$x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$$

The range of $\csc x$ is the same as that of $\sec x$, for the same reasons (except that now we are dealing with the multiplicative inverse of sine of x, not cosine of x). Therefore the range of $\csc x$ is

$$\csc x \geq 1 \qquad \text{or} \qquad \csc x \leq -1.$$

The period of $\csc x$ is the same as that of $\sin x$, which is 2π. Since $\sin x$ is an odd function, $\csc x$ is also an odd function. Finally, at all of the points where $\csc x$ is undefined, the function has both left and right vertical asymptotes, but just as in the case of $\sec x$, the behavior of the vertical asymptotes depends on the point. If $a = \ldots, -2\pi, 0, 2\pi, \ldots$, then the left vertical asymptote at a goes to negative infinity, and the right vertical asymptote goes to positive infinity:

$$\lim_{x \to a^-} \csc x = -\infty \qquad \text{and} \qquad \lim_{x \to a^+} \csc x = +\infty.$$

If, on the other hand, $b = -3\pi, -\pi, \pi, 3\pi, \ldots$, then the left vertical asymptote at b goes to positive infinity, and the right vertical asymptote goes to negative infinity:

$$\lim_{x \to b^-} \csc x = +\infty \qquad \text{and} \qquad \lim_{x \to b^+} \csc x = -\infty.$$

Derivatives of Trigonometric Functions

Now that we know the properties of all of the trigonometric functions, we should take their derivatives. All of the trigonometric functions are differentiable wherever they are defined. To find their derivatives, we use the quotient rule and the Chain Rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right) = \frac{(\cos x)\cdot(\cos x) - (\sin x)\cdot(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos x}{\sin x}\right) = \frac{(\sin x)\cdot(-\sin x) - (\cos x)\cdot(\cos x)}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cos x}\right) = -\frac{1}{\cos^2 x}\cdot(-\sin x) = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\csc x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sin x}\right) = -\frac{1}{\sin^2 x}\cdot(\cos x) = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} = -\csc x \cot x.
\end{aligned}$$

Now that we have these formulae for the derivatives of trigonometric functions, let us do some examples of using these formulae. Let $f(x) = \tan(1 + x^2)$. Then, where $f(x)$ is defined, we have that

$$f'(x) = \sec^2(1 + x^2)\cdot 2x = 2x\sec^2(1 + x^2).$$

As another example, let $g(x) = \mathrm{e}^{\sec x \tan x}$. Then, by the Chain Rule, we get that where $g(x)$ is defined its derivative is

$$g'(x) = \mathrm{e}^{\sec x \tan x}\cdot((\sec x \tan x)\cdot \tan x + \sec x \cdot \sec^2 x) = (\sec x \tan^2 x + \sec^3 x)\mathrm{e}^{\sec x \tan x}.$$

Finally, let $h(x) = \frac{1}{\sec x}$. Where $h(x)$ is defined, it should be equal to $\cos x$. Let us verify that, in the domain of h, the derivative of $h(x)$ is $-\sin x$:

$$h'(x) = -\frac{1}{\sec^2 x}\cdot \sec x \tan x = -\frac{\tan x}{\sec x} = -\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = -\sin x.$$

Technically, $\cos x$ and $h(x)$ are not the same function. Can you think of a reason why? (Hint: What are the domains of these functions?)