General Composition of Functions

The Composition of Two Functions

In the last lecture, we discussed the stretching and shifting of functions. Stretching and shifting are just two examples of a general phenomenon called the composition of functions. The composition of two functions is the topic for today's lecture.

Let $f(x)$ and $g(x)$ be two real valued functions whose domains are the real numbers. Polynomial functions and sine and cosine functions are examples of functions which fit this description. Suppose that we took a number x and applied the function f to it. We would get as our output the number $f(x)$. Suppose now that we applied the function g to the number $f(x)$. We would get another number as our output $g(f(x))$.

The key point for this section is that we can think of the action of starting with x, then applying f, then applying g, not just as two functions being applied one after another, but as one function, a composition of f and g. Our new function, which we write as $g \circ f$ and speak as "g compose f" or, perhaps more precisely, as "g after f," takes every real number x and gives us back the number $g(f(x))$.

You can think about the composition of functions in the following way: before, we described a function as a machine, which takes in raw material, its domain, and gives out a finished product, its range. Suppose that we have two such machines, one called f, the other called g. Suppose further that the raw material for g is the finished product of f. Then we could imagine setting up these two machines in an assembly line, so that as soon as the finished product comes out of the machine f, it goes into the machine g. How do we envision the composition $g \circ f$? What we do is imagine placing a big box around both the machine f and the machine g, so that when raw material goes into the machine f, we do not see it again until it comes out of the machine g. Now, if after we put this big box around f and g, someone looked at this big box and saw raw material going in and finished product coming out, he would assume that this big box was a single machine, even though we know that in reality it is two machines, working one after the other. That person would be perfectly correct, however, to assume that he is seeing only one machine, because it does not matter how the machine works inside: all that matters is that it has an input and an output. Now we, knowing that this big box encompasses two machines, want to label the box so as to inform people that this box contains the machine g working directly after the machine f. We do this by labelling the big box as $g \circ f$. It is still a single machine, but now people know how it works inside.

Let us do a few examples of composition using formulas instead of big boxes. Let $f(x) = \cos x$ and let $g(x) = x^3$. We get the formula for the composition $g \circ f$ by substituting the formula for $f(x)$ in for x in the formula for $g(x)$. In the case, we substitute $\cos x$ for x in $g(x) = x^3$, so that we get

$$(g \circ f)(x) = g(f(x)) = (f(x))^3 = (\cos x)^3.$$

Now let us try the composition $f \circ g$, the function we get by first applying g and then applying f. This time, we substitute x^3 in for x in $f(x) = \cos x$. The result is

$$(f \circ g)(x) = f(g(x)) = \cos(f(x)) = \cos(x^3).$$

This example illustrates a very important point about composition: usually, the function $g \circ f$ does not equal the function $f \circ g$. The order in which you compose two functions matters a great deal. For another example, take $f(x) = 2x$ and $g(x) = x^2$. Then

$$(g \circ f)(x) = g(f(x)) = (f(x))^2 = (2x)^2 = 4x^2,$$

and

$$(f \circ g)(x) = f(g(x)) = 2(g(x)) = 2x^2.$$

If you were to plot these two compositions alongside the graph of x^2, you would see that the first, $g \circ f$, is a horizontal stretch of the graph of x^2, while the second, $f \circ g$, is a vertical stretch of that parabola. This leads us into the next section.

Stretches and Shifts as Compositions

For more examples of composition, we study stretches and shifts of functions again. Remember that we stated that $g(x)$ is a vertical stretch of $f(x)$ if $g(x) = af(x)$. Let $h(x) = ax$. Another way to write the definition of a vertical stretch is to write $g(x)$ as a composition of $f(x)$ and $h(x)$. Specifically, $g(x)$ is a vertical shift of $f(x)$ if $g(x) = (h \circ f)(x)$ for some real number a. We see that this is true by finding the formula for $h \circ f$:

$$(h \circ f)(x) = h(f(x)) = af(x).$$

We defined $g(x)$ to be a horizontal stretch of $f(x)$ if $g(x) = f(ax)$ for some real number a. If we define $h(x) = ax$ again, then we quickly see that a horizontal stretch can also be understood to be a composition of two functions, specifically, $g(x) = (f \circ h)(x)$, since when we work out the formula for $f \circ h$, we get

$$(f \circ h)(x) = f(h(x)) = f(ax).$$

So, for a vertical stretch, we apply h to the range of f, and for a horizontal stretch, we apply h to the domain of f. This should make intuitive sense to you: The range of f corresponds to the vertical axis on the xy-plane, and the domain of f corresponds to the horizontal axis. Thus, if you want to change the graph of $f(x)$ horizontally, you want to act on its domain, and it you want to change its graph vertically, you act on its range.

We said that a function $g(x)$ is a vertical shift of $f(x)$ if for some real number b we have that $g(x) = f(x)+b$. How could we write $g(x)$ as the composition of two functions? One way to do it is to let $k(x) = x+b$. Then

$$(k \circ f)(x) = k(f(x)) = f(x) + b = g(x).$$

So g is the composition of k after f.

We defined $g(x)$ to be a horizontal shift of $f(x)$ if $g(x) = f(x + b)$ for some real number b. Clearly $g(x)$ is the composition of two functions in this case as well: setting $k(x) = x + b$ again, we see that

$$(f \circ k)(x) = f(k(x)) = f(x + b) = g(x).$$

Notice how for horizontal shifts k acts on the domain of f, and for vertical shifts, k acts on the range, just as in the case of stretches.

Seeing Functions as Compositions

In the previous section, we took known functions, the stretches and shifts of $f(x)$, and rewriting them as the composition of two functions. This is the most important skill that you must learn about compositions: recognizing that a function is the composition of two other functions. There are no simple rules for breaking up a function into the composition of two other functions; the best way to learn how this skill is to see a lot of examples.

First, let us take the example $h(x) = \sin^2 x + 9 \sin x + 8$. We see that this is a trigonometric polynomial, that is, a polynomial of $\sin x$ instead of x. This should suggest to you how to write this function as the composition of two other functions: the inside function, the one we do first, should be $\sin x$, and the outside function should be a polynomial. Specifically, take $f(x) = \sin x$ and $g(x) = x^2 + 9x + 8$. Then

$$(g \circ f)(x) = g(f(x)) = (f(x))^2 + 9f(x) + 8 = (\sin x)^2 + 9 \sin x + 8 = \sin^2 x + 9 \sin x + 8 = h(x).$$

For the next example, let $h(x) = (2x^2 + 3x - 7)^4$. This function is a polynomial raised to the fourth power. It stands to reason that, to write this function as the composition of two other functions, $g(x)$ after $f(x)$, the inside function, $f(x)$, should be the polynomial being raised to the fourth power, and the outside function, $g(x)$, should be x^4. So let $f(x) = 2x^2 + 3x - 7$ and let $g(x) = x^4$. Then

$$(g \circ f)(x) = g(f(x)) = (f(x))^4 = (2x^2 + 3x - 7)^4 = h(x).$$

So $h(x)$ is the composition of $g(x)$ after $f(x)$.

Notice the general principle at work here: we look for an inside function, a function being acted upon, and we let this function be $f(x)$. We then figure out how $f(x)$ is being acted upon, and we label the function that is doing the action by $g(x)$. Let us try two more examples.

Let $h(x) = \cos(x^{17})$. We can see that $\cos x$ is acting on x^{17}. So we let $f(x)$, the inside function, be equal to x^{17}, and we let $g(x)$, the outside function, be equal to $\cos x$. Then we get that

$$(g \circ f)(x) = g(f(x)) = \cos(f(x)) = \cos(x^{17}) = h(x),$$

so our choices of $g(x)$ and $f(x)$ work.

Finally, take $h(x) = \sin(\cos(x^2))$. This example presents us with a choice. First, we could take the inside function to be $f(x) = \cos(x^2)$ and the outside function to be $g(x) = \sin x$. Then

$$(g \circ f)(x) = g(f(x)) = \sin(f(x)) = \sin(\cos(x^2)) = h(x),$$

so this choice of $f(x)$ and $g(x)$ work. We could also have chosen the inside function to be x^2 and the outside function to be $\sin(\cos x)$. So let $m(x) = x^2$ and $n(x) = \sin(\cos x)$. Then we get that

$$(n \circ m)(x) = n(m(x)) = \sin(\cos(m(x))) = \sin(\cos(x^2)) = h(x).$$

So now we have two equally valid ways to break up $h(x)$ into the composition of two functions. You may also notice that we could see $h(x)$ as the action of three different functions, one after the other after the other. In this case, take $p(x) = x^2$, $q(x) = \cos x$, and $r(x) = \sin x$. Then we define the function $r \circ q \circ p$ to be a function we get by first acting on x by p, then acting on $p(x)$ by q, and then acting on $q(p(x))$ by r. This triple composition is equal to $h(x)$:

$$(r \circ q \circ p)(x) = r(q(p(x))) = \sin(q(p(x))) = \sin(\cos(p(x))) = \sin(\cos(x^2)) = h(x).$$

So now we have three different ways to break up $h(x)$ into the composition of two or more functions. We will use this skill in the next lecture when we discuss how to differentiate the composition of functions using the chain rule.