Products and Powers

Raising Functions to Positive Powers

During the last two lectures, we learned about our first non-polynomial functions: the sine and cosine functions. Today, and for the next few lectures, we will learn how to build new functions using polynomial and non-polynomial functions like sine and cosine. We begin today with raising functions to positive powers and with multiplying two functions together.

First, let us state the power rule of differentiation: suppose that $f(x)$ is a function with a derivative. Define $g(x)$ to be the function $f(x)$ multiplied by itself n times, where n is a natural number. Then $g(x)$ also has a derivative, and that derivative is given by

$$\frac{\mathrm{d}g}{\mathrm{d}x} = n(f(x))^{n-1}\frac{\mathrm{d}f}{\mathrm{d}x}.$$

There is a lot going on in this statement. First, we begin with a function with a derivative called $f(x)$. So, for example, we could have $f(x) = x^2 + \sin x$. This function has a derivative: $f'(x) = 2x + \cos x$. So, our first condition is satisfied.

Next, we define a new function, $g(x)$, which is $(f(x))^n$, that is, $f(x)$ multiplied by itself n times, where n is some natural number. For example, we could take $n = 5$. So, to continue our example from above, we get that $g(x) = (2x + \cos x)^5$. Note that there is no need to multiply out the various terms in this expression. That would take a long time, and for the power rule, it is not necessary.

Now we get to the heart of the power rule: the power rule tells us that $g(x)$, which, again, is $f(x)$ raised to the power of n, has a derivative itself. Moreover, the power rule gives us a formula for the derivative of $g(x)$, and that formula is

$$\frac{\mathrm{d}g}{\mathrm{d}x} = n(f(x))^{n-1}\frac{\mathrm{d}f}{\mathrm{d}x}.$$

To state this formula in words, the derivative of $g(x) = (f(x))^n$ is n times $f(x)$ multiplied by itself $n-1$ times, times the derivative of $f(x)$. So to take the derivative of $(f(x))^n$, we put the exponent out in front, drop the exponent by 1, and multiply by the derivative of $f(x)$. Let us try this with our example: we have that $g(x) = (x^2 + \sin x)^5$. By the power rule, $g(x)$ has a derivative, and that derivative is

$$\frac{\mathrm{d}g}{\mathrm{d}x} = 5 \cdot (x^2 + \sin x)^4 \cdot (2x + \cos x).$$

As another example, take the trigonometric polynomial $f(x) = \sin^2 x - 4\sin x + 3$. A trigonometric polynomial is a polynomial with x replaced by a trigonometric function. We know how to take the derivative of the second and third terms; to take the derivative of the first term, we notice that the first term is $\sin x$ raised to the power of 2, so we can apply the power rule to get its derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^2 x) = 2\sin x\left(\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)\right) = 2\sin x\cos x.$$

Now we use the sum rule to get the derivative of the whole trigonometric polynomial:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 2\sin x\cos x - 4\cos x.$$

Sometimes applications of the power rule are hidden. Take for example the function $f(x) = 9x^2 - 30x + 25$. We could take the derivative of this function very easily using our rule for taking the derivatives of quadratic functions, and we would get $18x - 30$ for the derivative. Looking closer, however, we see that $f(x)$ is the square of a linear function:

$$f(x) = 9x^2 - 30x + 25 = (3x - 5)^2.$$

Now, applying the power rule to this new formula for $f(x)$, we get the derivative

$$f'(x) = 2 \cdot (3x - 5) \cdot 3 = 18x - 30.$$

So the power rule and the quadratic rule are consistent. You may think that this application is a a waste of time, but consider the following: suppose you had to take the derivative of the following function:

$$f(x) = x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 8x + 1.$$

There are a lot of terms in this polynomial, so taking its derivative is a bit of work. You may recognize, however, that $f(x)$ is simply $(x+1)^8$, and if you do, the derivative is simple:

$$f'(x) = 8(x+1)^7.$$

You do not however have to know that $f(x) = (x+1)^8$ in order to take its derivative, because you could have simply used the techniques we have from taking the derivative of polynomials. The point is, when and if you use the power rule is up to you.

Finally, we want to finish this section by reminding you that we have used the power rule before this class: we used it when we were finding the derivatives of positive power functions. How do we find the derivative of x^n, where n is a natural number? We apply the power rule, with $f(x) = x$. Then $f'(x) = 1$, so the derivative of x^n is then

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1} \cdot 1 = nx^{n-1}.$$

So the power rule and the rule for taking the derivative of positive power functions are consistent with each other.

Products of Functions

Previously in this class, we found that, when we added two functions together, we got that the derivative of the sum is the sum of the derivatives. So you may think that, if we multiply two functions together, then the derivative of the product is the product of the derivatives. This is completely false.

The product rule, also called the Leibniz rule, for taking derivatives is the following: suppose that $f(x)$ and $g(x)$ are two functions with derivatives. Let $h(x) = f(x)g(x)$, the product of $f(x)$ and $g(x)$. Then $h(x)$ also has a derivative, and that derivative is given by

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Let us go through the product rule step by step. First, we need to start with two functions, each with derivatives. So, for example, we could take $f(x) = x^2$ and $g(x) = \sin x$. The derivatives of these two functions are $f'(x) = 2x$ and $g'(x) = \cos x$. Next, we multiply these two functions together to get $h(x)$. In this case, we get that $h(x) = x^2 \sin x$.

Now we apply the power rule, which tells us that $h(x)$ also has a derivative, and we can find it using the formula

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

We usually verbally state this formula by saying "derivative of the first times the second plus derivative of the second times the first." Notice the symmetry in the formula, and how we "distribute" out the derivative one function at a time.

Now, to finish our example, we get that the derivative of $h(x) = x^2 \sin x$ is

$$h'(x) = 2x \cdot \sin x + x^2 \cdot \cos x = 2x \sin x + x^2 \cos x.$$

Why do we need the product rule? Consider positive power functions, like $f(x) = x^5$. We know what the derivative of $f(x)$ is: using the power rule, we get that $f'(x) = 5x^4$. Now, notice that we could also write $f(x)$ as a product. One way of doing this is $f(x) = x^3 \cdot x^2$. Applying the product rule here, we get that

$$f'(x) = 3x^2 \cdot x^2 + x^3 \cdot 2x = 3x^4 + 2x^4 = 5x^4,$$

which is precisely what we expected to happen. Now suppose that, instead of using the product rule correctly, we simply multiplied the derivatives together to get the derivative of the product. We would get that $f'(x) = 3x^2 \cdot 2x = 6x^3$, which is completely false. So, somehow, the power rule and the product rule are consistent with each other. Keep this example in mind so that you do not make the mistake of thinking that the derivative of the product is the product of the derivatives.

Another check of the product rule is the following: take any function with a derivative, like $f(x)$. We can write this function as a product of itself and the constant function 1:

$$f(x) = 1 \cdot f(x).$$

The derivative of any constant function is 0, so if it were true that the derivative of the product equalled the product of the derivatives, then, in this case, $f'(x) = 0 \cdot f'(x) = 0$, which is ridiculous. Applying the real product rule, we get the right answer:

$$f(x) = 0 \cdot f(x) + 1 \cdot f'(x) = 0 + f'(x) = f'(x).$$

In fact, we can use the product to confirm our formula for the derivative of $f(x) = 1$. We write $f(x)$ as a product of 1 with itself: that is, $f(x) = 1 \cdot 1$. Now we apply the product rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(1) &= \frac{\mathrm{d}}{\mathrm{d}x}(1 \cdot 1) \\
\frac{\mathrm{d}}{\mathrm{d}x}(1) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}(1)\right) \cdot 1 + 1 \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x}(1)\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(1) &= 2\left(\frac{\mathrm{d}}{\mathrm{d}x}(1)\right) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}x}(1).
\end{aligned}$$

The product rule forces the derivative of $f(x) = 1$, and all constant multiples of $f(x) = 1$ (that is, all constant functions) to be 0.

We finish this lecture by applying both the power rule and the product rule to find the second derivative of $f(x) = \sin^2 x$. First, we find the first derivative, as we did before, using the power rule:

$$f'(x) = 2 \cdot \sin x \cdot \cos x = 2 \sin x \cos x.$$

Next, we use the product rule to find the derivative of the derivative:

$$f''(x) = 2\cos x \cdot \cos x + 2 \sin x \cdot (-\sin x) = 2\cos^2 x - 2\sin^2 x = 2(\cos^2 x - \sin^2 x).$$

The power rule and the product rule are indispensable to calculus, and you need to know how to use them properly in order to succeed in this class.