

Functions

Mathematical Modeling

Why do we learn mathematics? One answer: to make the world a little easier to understand.

How does knowing mathematics make the world easier to understand? Consider the process of mathematical modeling. Mathematical modeling is the process by which we reduce a difficult real world problem to a somewhat simpler mathematical object using physical laws and common sense. We then draw mathematical conclusions from that mathematical object, using logic and techniques like the ones we will learn in this class. We apply these mathematical conclusions to the real world to get real world predictions. We then test those predictions using scientific experiments and the scientific method. If those predictions hold up under scientific scrutiny, we say that we have a mathematical model of our real world problem, and we have successfully made the world easier to understand. If the predictions do not hold up, we refine our mathematical problem, and we repeat the process until we have a mathematical model with which to understand reality.

In this course, you will learn about one type of mathematical object, the real-valued function, and you will learn about single-variable differential calculus, a set of techniques used to draw conclusions about those real-valued functions. Hopefully, by the end of this class, you will be well-prepared to use these functions and calculus to model real world problems in whatever field you decide to study.

The Real Numbers

The main way that we model real world problems with mathematics is by converting information which we have about that problem into numbers. The first numbers with which we became familiar are the natural numbers, or counting numbers. They are $1, 2, 3, 4, \dots$. These are the numbers that we learned about by counting our fingers and our toes. At some point in the last two thousand years, the Mayans and the Hindus independently realized that the natural numbers were not enough numbers to understand reality, so they invented the number 0, a way of representing nothingness.

You can represent the natural numbers and 0 together on a line by placing a point on the line to represent 0, and then, moving to the right, by placing one point after another at equal distances to represent the numbers 1, 2, 3, and so on. If our line went out an infinite distance from 0 to the right, we could represent all of the natural numbers in this way.

Looking at this line, it is easy to see how we could make more numbers: we could place points on after another at equal distances to the left. In this way we get negative numbers: the first point would be called -1 , the second -2 , and so on. The numbers now represented by points on this line, that is, the natural numbers, the number 0, and the numbers $-1, -2, -3, -4, \dots$, are together called the integers.

We know a little about the integers: we know how to add them together, how to subtract one from another, and how to multiply them. We know that the sum, difference, and product of two integers is also an integer. Unfortunately, when we try to divide one integer by another, say $14 \div 10$, we find the quotient is not an integer at all. It is a fraction, in this case $\frac{7}{5}$. We can represent $\frac{7}{5}$ on our line by taking the space between the point 0 and the point 1, breaking it up into five equal size spaces with four new points, and marking the third new point as $\frac{3}{5}$. We can use this method to plot every fraction, and in doing so we seem to fill up the entire number line with points. The fractions, together with the integers (which can be fractions themselves), are called the rational numbers, and this is how we see them.

The points which represent the rational numbers seem to fill up the number line, but this is not the case: in fact, we have missed almost every possible point we could have marked on the line. The missing points are called the irrational numbers. There are two irrational numbers which will be very important in this class: $\pi = 3.141592654\dots$ and $e = 2.718281828\dots$. You do not need to know the specific values of these numbers: just know that they exist and that they cannot be written as fractions. The rational numbers and the irrational numbers together are called the real numbers. The real numbers form a continuum, that is, we can represent them as a line without any gaps. We call this line the real number line, and it is our mathematical model for time, distance, temperature, mass, and a myriad of other real world phenomena. This is the setting for all of the work we will do in this class.

Functions

Often times, real world problems involve associating one quantity which can be modelled by the real line with another quantity which can be modelled by the real line. For example, we can associate a time with the temperature in Hanover at that time. This association is called a function, and specifically, a real valued function, since the quantities we are associating, time and temperature, are both modelled by the real line.

In the real world, there may be many reasons why one quantity would be associated to another: the temperature in Hanover may change over time because the sun rises and sets, because the Earth is moving around the sun, because the temperature in Hanover slightly increases when there are 5000 students on campus. The beauty of functions is that we do not need to understand how we associate one quantity to another: all we need to know is that there is an association and what that association is. We do not need to know why the temperature of Hanover was -27 degrees Fahrenheit on February 17, 2003 at 5:26 A.M.: we just need to know that Hanover had a temperature on February 17, 2003 at 5:26 A.M., and that temperature was -27 degrees Fahrenheit.

You can think of a function as a machine which turns some raw material into a finished product. You do not need to know how the machine works to know that it works. The raw material of a function is called its domain. The finished product of a function is called its range. For real valued functions, the domain is the real numbers or some subset of the real numbers, and the range is also the real numbers or some subset of them. Try to keep this machine analogy in mind when you think about functions, as this is a very accurate way of thinking about this mathematical concept.

Representing Functions

So far, we know of one good way to represent a function: we can describe it verbally. Thus “the temperature of Hanover as a function of time” is a perfectly good representation of that function. On the other hand, it is not very useful for mathematical purposes, because we cannot draw any conclusions about the temperature of Hanover at a given time using that representation.

A more powerful representation is the algebraic, or symbolic, representation of a function. We represent the input of a function by a variable, for example, t for time. We represent the function by another letter, often times f , and we represent the output of the function by $f(t)$, which is spoken as “ f of t .” Thus in our previous example, $f(t)$ would be the temperature of Hanover at time t .

This representation is not very useful unless we can say that $f(t)$ is equal to some algebraic expression. Moving away from our previous example, let us consider the function which associates to a square with sides of length x the area of that square. We will call this function A , so the area of the square with side length x is $A(x)$. (Incidentally, this is an example of a function whose domain is not all of the real numbers, since no square can have sides with negative length.) We all know that a square with side length x has area x^2 , so we write $A(x) = x^2$.

This new representation is very useful from a mathematical standpoint. We can use it to derive a third representation of a function, that being a numerical table. A numerical table is a table which lists a set of values of the domain alongside the values of the function at those particular points in the domain. For example, we can represent the function A by a numerical table by listing the numbers 0, 1, 2, 3, and 4 alongside the squares of those numbers:

x	$A(x)$
0	0
1	1
2	4
3	9
4	16

A numerical table is not meant to show you the value of the function for all elements of the domain, and in fact it is impossible to do that in most cases, even if you had an infinite amount of time to construct your table. A numerical table can be very useful, however, if you want to see how the function behaves close to some number, or as x gets bigger and bigger. We will learn more about this as the term goes on.

Perhaps the most powerful way to represent many functions is graphically. In a graph, we represent all of the possible values that a function can have by a plane. We represent the largest possible domain as a horizontal line, usually called the x -axis, and the largest possible range is represented by a vertical line, usually called the y -axis. The variable x is called the independent variable, and the variable y is called the dependent variable. The value of x increases as one moves from left to right; the value of y increases from bottom to top. The points on the x -axis have a y value of 0, the points on the y -axis have an x value of 0, and the point where the two axes cross, the origin, has both x and y value of 0. We write a point on this plane as an ordered pair (x, y) , with the x value of the point first and the y value second. Thus points on the x -axis are of the form $(x, 0)$, points on the y -axis are of the form $(0, y)$, and the origin is the point $(0, 0)$. We call this description of a point its coordinates.

The graph of a function is (usually) a curve, where a point (x, y) is on the curve if and only if $y = f(x)$. We can use a numerical table of a function to help draw its graph: we simply plot the point $(x, f(x))$ for each value of x we have in the table, and then draw a curve which runs through all of those points. Unfortunately, given a numerical table for a function, there are an infinite number of curves which can pass through the points plotted from that table, and only one of them can be the graph of the function. For certain functions, however, we can use the techniques of calculus to narrow down the possibilities, and you will learn how to do this throughout the term.

The Vertical Line Test

For every value in the domain of a function, the function can take exactly one value. For example, Hanover (or at least a specific point in Hanover) cannot be two temperatures at once. This is a fundamental characteristic of all functions. Because of this characteristic of functions, not every curve on the xy -plane can be the graph of a function. For example, if we draw a circle of radius 1 centered at the origin, this could not be the graph of a function, because there are two points, $(0, 1)$ and $(0, -1)$, where the x coordinate is 0, which would mean that if this circle was the graph of the function f , then $f(0) = 1$ and $f(0) = -1$, which is absurd.

An easy way to determine if a curve on the xy -plane cannot be the graph of a function is to draw vertical lines through the curve for various values of x . If any of the vertical lines cross the curve more than once, then the curve cannot be the graph of a function. This is because the places where the vertical line crosses have the same x -value, which means that if that curve were the graph of a function f , then for that value of x , $f(x)$ would have more than one value. This method is called the Vertical Line Test, and you should understand why it can tell you that a curve is not the graph of a function.