Variance when variables are not independent

Consider the experiment of rolling two dice, and let X on each roll be the value showing on the first die, and Y on each roll be the sum of the values showing on both dice. As we have seen before,

$$\begin{array}{ccc} 1 \le X \le 6 & E(X) = 3.5 & V(X) = 2.92 \\ 2 \le Y \le 12 & E(Y) = 7 & V(Y) = 5.84 \end{array}$$

X and Y are not independent: $p(X = 1)$ and $p(Y = 12)$ are both nonzero, but $p(X = 1 \mathbin{\&} Y = 12) = 0 \ne p(X = 1) \cdot p(Y = 12)$.

Consider all the values of Y and $X + Y$ in a table where die 1's value is across the top and die 2's value down the left side (values of X are the column headers):

Y, $X+Y$	1	2	3	4	5	6
1	2, 3	3, 5	4, 7	5, 9	6, 11	7, 13
2	3, 4	4, 6	5, 8	6, 10	7, 12	8, 14
3	4, 5	5, 7	6, 9	7, 11	8, 13	9, 15
4	5, 6	6, 8	7, 10	8, 12	9, 14	10, 16
5	6, 7	7, 9	8, 11	9, 13	10, 15	11, 17
6	7, 8	8, 10	9, 12	10, 14	11, 16	12, 18

You can see in this that overall, higher values of X tend to go with higher values of Y and lower with lower; this makes the distribution of values of $X + Y$ flatter (more pushed to the highest and lowest values) than it would be if X and Y were independent (say, we had three dice and Y was the sum on dice 2 and 3). In particular:

$p(X + Y = k) = 1/36$ for $k \in \{3, 4, 17, 18\}$
$p(X + Y = k) = 1/18$ for $k \in \{5, 6, 15, 16\}$
$p(X + Y = k) = 1/12$ for $k \in \{7, 8, 9, 10, 11, 12, 13, 14\}$.

You can check that $E(X + Y) = 10.5 = E(X) + E(Y)$; we don't need independence for adding expected values. We do need independence for multiplying expected values, which is the issue when we consider variance.

To find $V(X + Y)$ directly, find the expected value of $(X + Y)^2$:

$$\begin{aligned} E((X+Y)^2) &= \frac{9 + 16 + 289 + 324}{36} + \frac{25 + 36 + 225 + 256}{18} \\ &+ \frac{49 + 64 + 81 + 100 + 121 + 144 + 169 + 196}{12} \\ &= \frac{638}{36} + \frac{542}{18} + \frac{924}{12} = 124.83 \end{aligned}$$

$$V(X + Y) = E((X + Y)^2) - E(X + Y)^2 = 124.83 - 110.25 = 14.58$$

On the other hand, $V(X) + V(Y) = 2.92 + 5.84 = 8.76$, which is smaller, as expected.