Math 19 final exam review problems

Note: these cover a lot of material but are not guaranteed to be either comprehensive, or representative of the relative proportions of the material on the exam.

(1) Give an example of functions f_1, f_2, g such that f_1 and f_2 are each $\Theta(g)$ but $f_1 + f_2$ is not $\Theta(g)$.

(2) A computer program randomly orders the cards in a standard 52-card deck, and then reports to the user the sequence of suits, without the values. How many sequences are possible? What if it reports the values without the suits?

(3) Find the value of the sum $\displaystyle\sum_{k=10}^{20}\sum_{j=4}^{12}(j^2 2^k + 10)$.

(4) What is wrong with the following proofs? You should be able to sum each up in a single sentence.

Claim. For every positive integer n, $n^2 + n$ is odd.

Proof. We work by induction. First note the number $n = 1$ is odd. Now suppose for some integer $n \geq 1$, $n^2 + n$ is odd. Then

$$(n+1)^2 + (n+1) \;=\; n^2 + 2n + 1 + n + 1 \;=\; (n^2+n) + (2n+2),$$

which is the sum of an odd number and an even number, and hence odd. Therefore by induction, $n^2 + n$ is odd for all positive integers n. □

Claim. For each natural number n there is a natural number greater than n^2.

Proof. Let n be a natural number, say 5. Then $n^2 = 25$ is a natural number, and 26 is a natural number larger than 25. □

Claim. Define the relation R on $\mathbb{Z}$ by $R(n,m) \leftrightarrow [n - m \text{ is even}]$. Then R is an equivalence relation.

Proof. Suppose $R(n,m) \leftrightarrow [n-m \text{ is even}]$ is an equivalence relation. Then it is reflexive; that is, for all $n \in \mathbb{Z}$, $R(n,n)$ holds. R is also symmetric: if $R(n,m)$ holds, so $n - m$ is even, then $R(m,n)$ holds and $m - n$ is also even. Finally, R is transitive: if $R(n,m)$ and $R(m,p)$ hold, so $n - m$ and $m - p$ are even, then $R(n,p)$ also holds and $n - p$ is even. □

Claim. For real numbers a and b, $-b \leq a \leq b$ if and only if $|a| \leq b$.

Proof. Suppose that a and b are such that $-b \leq a \leq b$. If a is zero, clearly $|a| \leq b$. If a is positive, then $0 < a \leq b$ and $|a| \leq b$; if a is negative, then $-b \leq a < 0$, so $0 < -a \leq b$ and again $|a| \leq b$. □

(5) Count each of the following situations with the appropriate permutation or combination.

   (a) Joan's test contains a matching problem, with six different vocabulary words and eight definitions. One definition will be matched to each word and there will be two left over. In how many ways could Joan answer the test problem?

   (b) A grade school music teacher needs to pick 8 of her 25 recorder players to play in a retirement home concert. Assuming she has no preference as to which students go, in how many ways can she make the selection?

   (c) A class of 32 students is going to do a project. There are two versions of the project available, version A and version B. If the projects are assigned at random, in how many assignments will 14 students do project A and 18 do project B?

   (d) There are fifteen raffle tickets numbered one through fifteen, and each corresponds to a different prize. If 150 people bought raffle tickets, and no one can win more than one prize, in how many ways may the winners be chosen?

(6) Suppose you need to know C(21, 7). How can you use the following pieces of information to give you the answer? Try to find as many pathways as possible; you may not use all the information given.

P(21, 7)	P(21, 14)	7!
C(21, 14)	C(14, 7)	14!

(7) A factory has three assembly lines, I, II, and III, and each produces a certain amount of the total output as shown below. In addition, each line produces a certain amount of defective material. An item is chosen at random and found to be defective. Given the information in the table below, calculate the probability that this defective item was produced by line III.

Line	% of total output	% defective
I	60%	1%
II	30%	2%
III	10%	3%

(8) For each of the following relations on the set A, (if not already given) give the corresponding subset of $A \times A$ and a formula describing the relation. If A is finite, give a graphical representation of the relation.

   (a) $A = \{1, 2, 3, 4\}$. R_1 the relation given by the identity function. $R_2(x, y) \leftrightarrow x < y$. $R_1 \cup R_2$. $R_1 \cap R_2$.

   (b) $A = \mathbb{Z}$. $R_1(x, y) \leftrightarrow |x| = |y|$. $R_2(x, y) \leftrightarrow x^2 = y^2$. $R_1 \cup R_2$. $R_1 \cap R_2$.

   (c) $A = \{a, b, c, d\}$. $R_1 = \{(a, b), (b, c), (c, d), (d, a)\}$. $R_2 = \{(a, c), (c, a), (b, d), (d, b)\}$. $R_1 \cup R_2$. $R_1 \cap R_2$.

(9) Give a shorter, but equivalent, description of each of the following relations.

(a) R_- on $\mathcal{P}(\mathbb{N})$ where $R_-(A, B) \leftrightarrow A - B = \emptyset$.

(b) $R_{(\cap)}$ on $\mathbb{R}$ where $R_{(\cap)}(x, y) \leftrightarrow (-\infty, x) \cap (y, \infty) = \emptyset$.

(c) $R_{[\cap]}$ on $\mathbb{R}$ where $R_{[\cap]}(x, y) \leftrightarrow (-\infty, x] \cap [y, \infty) = \emptyset$.

(d) $R_{(\cup)}$ on $\mathbb{R}$ where $R_{(\cup)}(x, y) \leftrightarrow (-\infty, x) \cup (y, \infty) = \mathbb{R}$.

(e) $R_{[\cup]}$ on $\mathbb{R}$ where $R_{[\cup]}(x, y) \leftrightarrow (-\infty, x] \cup [y, \infty) = \mathbb{R}$.

(10) There are eight standard classifications for blood type (e.g., AB^-, O^+). A lab technician must take an exam wherein he or she determines the type of three blood samples.

(a) If the samples must all have different types, how many selections of three blood types are possible?

(b) If the examinee knows all three samples will be different, what is the probability that he or she will guess the correct set of three types?

(c) If the examinee knows all three samples will be different, what is the probability he or she will guess the correct set of types and match each to the correct sample?

(11) In Texas Hold 'Em, a variety of poker, each player is dealt two cards and then three are dealt face-up on the table. A standard 52-card deck is used.

(a) What is the probability player 1 is dealt a pair of aces?

(b) What is the probability player 1 is dealt a pair of aces *and* at least one ace is laid face up on the table, giving player 1 three or four of a kind?

(c) What are the probabilities for parts (a) and (b) if "ace" is replaced by "king"?

(d) If the deck is thoroughly shuffled after every game, what is the probability of player 1 getting three or four of a kind in both of the first two games?

(12) The Venusian sociologist studying embarrassing behaviors is trying a new method. Each Venusian participating in her study will get a survey asking "do you blizzum?" and a coin. They will enter a private booth and flip the coin. If it shows heads, they are to answer "yes". If it shows tails, they are to answer honestly. Experience suggests that because "yes" answers are so likely to be from the coin flip, everyone taking the survey will be honest. Assume that Venusians get a truly random flip.

(a) Suppose the actual proportion of Venusians who blizzum is $\frac{1}{25}$. If a Venusian answered "yes" on the survey, what is the probability that it is because the coin showed heads?

(b) Now suppose the actual proportion of blizzuming Venusians is unknown. If the sociologist's survey turns up 52% "yes" answers, what should she calculate the proportion to be?

(13) Find $E(X)$ and $V(X)$ for X with the following probability distribution.

k	$p(X = k)$
-1	.25
0	.1
1	.25
2	.4

(14) If $p(E) = 0.4$, $p(F) = 0.7$, and $p(E \cap F) = 0.25$, are E and F independent?

(15) Find the value of the sum $\displaystyle\sum_{i=2}^{10}\sum_{j=1}^{15}(20i - \frac{1}{2}j^3)$.

(16) Suppose an equivalence relation R on a set of size kn has k equivalence classes of n elements each. How many edges are in the directed graph representing R?

(17) A tennis group has 20 members, who are to be paired off for singles matches.

(a) Suppose first we have 10 numbered tennis courts. How many ways can we assign a pair to each tennis court?

(b) Suppose in addition to assigning the court, we also decide who serves first. Now how many ways are there to specify the pairs?

(c) Finally, determine how many ways there are to pair off the 20 people without specifying either court number or first person to serve.

(18) Consider hands drawn from a standard deck of 52 cards, with four suits of 13 cards each. How many five-card hands contain cards from exactly two suits?

(19) The Lucas numbers are very similar to the Fibonacci numbers, but have different starting values. Define them as follows, noting they start with index 1 instead of 0:

$$L_n = \begin{cases} 1 & n = 1 \\ 3 & n = 2 \\ L_{n-1} + L_{n-2} & n > 2 \end{cases}$$

Prove that for all $n \geq 1$, $L_1^2 + L_2^2 + \ldots + L_n^2 = L_n L_{n+1} - 2$.

(20) Prove that x^2 is $O(\frac{1}{100}x^3)$ but not $O(100x)$.

(21) Find domains of quantification that make each of the following formulas true or false.

(a) $\exists x \forall y(x + y = 10)$

(b) $\forall y \exists x(x + y = 10)$