Math 17: Knot Theory

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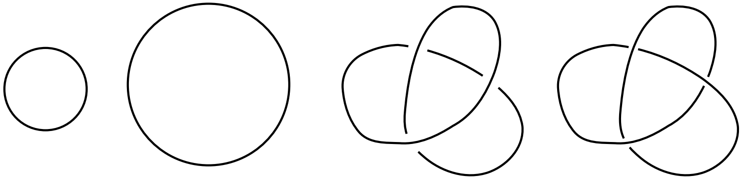
What is a knot?

Informally, you can think of a mathematical *knot* as a piece of string that is tied up with its two ends connected (we think of the string as having no thickness). We say that two knots are *equivalent* (or *the same*) if we can stretch and move one of them around in space, without breaking it, until it coincides with the other one.

Examples



The first three knots are the same (it's easy to deform one into another) and are called the *unknot*. The third appears to be different (in fact, we'll soon see that it is, and it's called the *right-handed trefoil*).

History

Why would anyone be interested to study knots?

Well, way back, in the 1800s, Lord Kelvin had a theory that atoms were knots in space, and that properties of the elements were related to knotting between atoms. And so Tait decided to try to make a table of knots... At the time, the table only had knots of up to 10 crossings, and there was no way of proving whether two knots are the same or not. Soon after, Kelvin turned out to be incorrect, but by then, knot theory was already growing as its own field of math.

History

In the early 20^{th} century, mathematicians developed some algebraic and other methods to distinguish knots. We'll focus on some of these methods over the next couple of weeks. Nowadays, knot theory is a very hot field for research, as knots are a great way to study 3- and 4-dimensional spaces. Ironically, knots have recently reappeared in chemistry (knotting of DNA is important to understand)!

More formally...

You can define knots more formally, by using equations, for example.

Example 1: $\{(x, y, z) \mid x^2 + y^2 = 1, z = 0\}$

Example 2: $\{(x, y, z) \mid y^2 + z^2 = 5, x = 4\}$

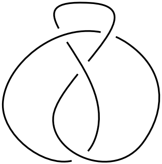
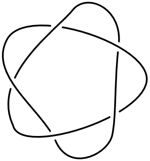
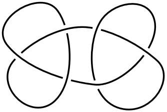
Both are a trivial knot.

More formally...

It's most convenient to work with pictures on a plane, instead of curves in $\mathbb{R}^3$. Given a knot K in $\mathbb{R}^3$, its *projection* is the image of K under the projection map to $\mathbb{R}^2$

$$(x, y, z) \to (x, y)$$

For example, a projection could look something like this.

More formally...

By moving the knot around a little, we can always arrange that its projection doesn't have triple points or tangencies. We'll only work with such nice projections from now on.

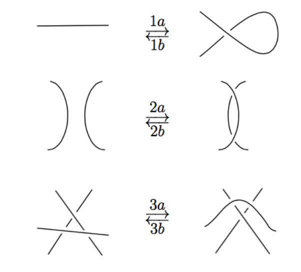
We also want to keep track of which strand goes over the other one at a crossing, so we leave little gaps to indicate this. The result is called a *knot diagram*.

When are two knots the same?

One big question in knot theory is whether two knots are the same or different.

Theorem

If two knots are the same, then their diagrams are related by the sequence of the following moves, called the Reidemeister moves:



When are two knots the same?

We won't prove this theorem (it's not super hard though), but we're free to use it from now on.

With a sequence of two Reidemeister moves, we transformed the third diagram from the beginning of these notes into the first diagram.

Reidemeister moves are cool, because if we ever need to prove something about a knot, we would only need to prove it for a diagram, and show that it remains true if we perform one of the 6 Reidemeister moves. This is much easier than having to prove something infinitely many times, once for each of the infinitely many diagrams for a given knot!

Knot colorings

So, we still can't show that the fourth knot is different from the first three. Here's our first tool that can distinguish knots:

Definition

A knot diagram is *colorable* if each arc can be drawn using one of three colors (say blue, red, and green) so that

1. at least two colors are used,
2. at each crossing, either only one color is used, or all three colors are used.

Knot colorings

We colored the fourth knot above by using a different color for each of its three arcs. We couldn't color the first two knots, since they only consist of one arc each, so Condition (1) can't be satisfied. We also saw that we can't color the third knot. That raised the question, "Is there any diagram for the unknot at all which is colorable?"

It turns out that:

Theorem

If one diagram of a knot is colorable, then every diagram for that knot is colorable.

Using Reidemeister moves, we can prove this theorem! Think about this before next class...