Math 17
Winter 2015
Exercises due Friday, February 27

We are showing in class that

$$\{(a, b, c) \mid b \geq 4 \ \& \ a = \alpha_b(c)\}$$

is a Diophantine set. This homework set will reproduce the proof from the textbook, from this fact, that exponentiation is Diophantine, asking you to fill in the details.

The goal is to show that

$$a = b^c$$

can be rewritten in a Diophantine way, as

$$[b = 0 \ \& \ c = 0 \ \& \ a = 1] \ \vee \ [b = 0 \ \& \ c > 0 \ \& \ a = 0] \vee$$

$$[b > 0 \ \& \ (\exists x) \ [x \geq 4 \ \& \ x > 16(c+1)\alpha_{b+4}(c+1) \ \& \ a = \alpha_{bx+4}(c+1) \ div \ \alpha_x(c+1)]].$$

Note that this expression is Diophantine, because all the subscripts of α are at least 4, and we are showing that for $b \geq 4$ we can express $a = \alpha_b(c)$ in a Diophantine way.

To meet this goal, we need to show that if $b > 0$ and

$$x \geq 4 \quad \& \quad x > 16(c+1)\alpha_{b+4}(c+1)$$

then

$$b^c = \alpha_{bx+4}(c+1) \ div \ \alpha_x(c+1).$$

From now on, we assume $b > 0$.

Exercise 1: Prove by induction on n that, for every natural number n,

$$(b-1)^n \leq \alpha_b(n+1) \leq b^n.$$

Exercise 2: Show that

$$\frac{\alpha_{bx+4}(c+1)}{\alpha_x(c+1)} \geq \frac{(bx+3)^c}{x^c} \geq b^c$$

Now we have shown that

$$b^c \leq \alpha_{bx+4}(c+1) \ div \ \alpha_x(c+1).$$

All that remains is to show that if

$$x > 16(c+1)\alpha_{b+4}(c+1)$$

then

$$b^c \geq \alpha_{bx+4}(c+1)\ div\ \alpha_x(c+1).$$

Exercise 3: Show that

$$16(c+1)\alpha_{b+4}(c+1) \geq 16(c+1)(b+1)^c.$$

Hint: Use Exercise 1.

This shows that

$$x > 16(c+1)\alpha_{b+4}(c+1) \implies x > 16(c+1)(b+1)^c,$$

so now, we will be done if we can show that if

$$x > 16(c+1)(b+1)^c$$

then

$$b^c \geq \alpha_{bx+4}(c+1)\ div\ \alpha_x(c+1).$$

So suppose $x > 16(c+1)(b+1)^c$. Therefore $x > 16c$, and we have

$$\frac{\alpha_{bx+4}(c+1)}{\alpha_x(c+1)} \leq \frac{(bx+4)^c}{(x-1)^c} \leq \frac{(1+\frac{4}{x})^c}{(1-\frac{1}{x})^c}\,b^c \leq \frac{b^c}{(1-\frac{1}{x})^c(1-\frac{4}{x})^c} \leq \frac{b^c}{(1-\frac{4}{x})^{2c}} \leq \frac{b^c}{1-\frac{8c}{x}} \leq b^c\left(1+\frac{16c}{x}\right)$$

Each inequality in this chain, of course, must be justified. Here is a justification for the last one:

Since $x > 16c$, we have $\frac{1}{x} < \frac{1}{16c}$; multiplying by $8c$ gives $\frac{8c}{x} < \frac{1}{2}$, so $1 - \frac{8c}{x} > \frac{1}{2}$, so $\frac{1}{1-\frac{8c}{x}} < 2$. Multiplying by $\frac{8c}{x}$ gives $\frac{\frac{8c}{x}}{1-\frac{8c}{x}} < 2\left(\frac{8c}{x}\right) = \frac{16c}{x}$, so $1 + \frac{\frac{8c}{x}}{1-\frac{8c}{x}} < 1 + \frac{16}{x}$, and

$$\frac{b^c}{1-\frac{8c}{x}} = b^c\left(\frac{1}{1-\frac{8c}{x}}\right) = b^c\left(\frac{1-\frac{8c}{x}+\frac{8c}{x}}{1-\frac{8c}{x}}\right) = b^c\left(1+\frac{\frac{8c}{x}}{1-\frac{8c}{x}}\right) < b^c\left(1+\frac{16}{x}\right).$$

Exercise 4: Explain why each remaining inequality in the chain is true.

Exercise 5: Show that it follows from

$$b^c \leq \frac{\alpha_{bx+4}(c+1)}{\alpha_x(c+1)} \leq b^c\left(1 + \frac{16c}{x}\right)$$

and

$$x > 16(c+1)(b+1)^c$$

that

$$b^c = \alpha_{bx+4}(c+1) \ div \ \alpha_x(c+1).$$

Since

$$b^c \leq \frac{\alpha_{bx+4}(c+1)}{\alpha_x(c+1)}$$

was shown in Exercise 2, and

$$\frac{\alpha_{bx+4}(c+1)}{\alpha_x(c+1)} \leq b^c\left(1 + \frac{16c}{x}\right)$$

was shown in Exercise 4, this completes the proof that exponentiation ins Diophantine.