Limits of Polynomials at Infinity

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Abstract

This paper is an example of an expository mathematics paper, illustrating appropriate format and style. It presents some basic facts about the limit of a polynomial $P(x)$ as x approaches $\pm\infty$.

1 Introduction

In this paper, we look at the limit of a polynomial $P(x)$ as x approaches $\pm\infty$. Specifically, we show that if the degree of $P(x)$ is at least 1, and the leading coefficient of $P(x)$ is a, then

$$\lim_{x\to\infty} P(x) = \begin{cases} \infty & \text{if } a > 0; \\ -\infty & \text{if } a < 0. \end{cases}$$

and

$$\lim_{x\to-\infty} P(x) = \begin{cases} \begin{cases} \infty & \text{if } a > 0, \\ -\infty & \text{if } a < 0, \end{cases} & \text{if } n \text{ is even;} \\ \\ \begin{cases} -\infty & \text{if } a > 0, \\ \infty & \text{if } a < 0, \end{cases} & \text{if } n \text{ is odd.} \end{cases}$$

2 Limits

According to Definition 3.4.6 of Stewart's calculus text[1],

$$\lim_{x\to\infty} f(x) = \infty$$

means

> for every positive number M there is a corresponding positive number N such that if $x > N$ then $f(x) > M$.

It is worth noting that "if $x > N$ then $f(x) > M$" means that for *every* x such that $x > N$, we have $f(x) > M$.

(See Section 3 for an explanation of this definition.)

This means that, in order to show that $\lim_{x\to\infty} P(x) = \infty$, we need to show that for every (positive) number M, we can find a corresponding (positive) number N such that

$$x > N \implies f(x) > M.$$

The definitions for limits involving $-\infty$ are as follows:

$$\lim_{x\to\infty} f(x) = -\infty$$

means

> for every positive number M there is a corresponding positive number N such that if $x > N$ then $f(x) < -M$;

$$\lim_{x\to-\infty} f(x) = \infty$$

means

> for every positive number M there is a corresponding positive number N such that if $x < -N$ then $f(x) > M$;

$$\lim_{x\to-\infty} f(x) = -\infty$$

means

> for every positive number M there is a corresponding positive number N such that if $x < -N$ then $f(x) < -M$.

3 About the Definition of Limit

In this section, we see how to arrive at the definition from Stewart[1] given in Section 2. The reader who is comfortable with this definition may skip this section. Here is the definition:

$$\lim_{x\to\infty} f(x) = \infty$$

> means for every positive number M there is a corresponding positive number N such that if $x > N$ then $f(x) > M$.

Intuitively, $\lim_{x\to\infty} f(x) = \infty$ means that as x gets larger and larger, so does $f(x)$. A little more precisely:

You can make $f(x)$ as large as you like, by making x large enough.

We will be more precise by phrasing the notion "large" as "greater than some chosen number."

First we will be more precise about "as large as you like":

For any number M, you can make $f(x)$ larger than M by making x large enough.

There is no harm in specifying that M be positive, because, for example, you can certainly make $f(x)$ larger than -2 if you can make $f(x)$ larger than 1.

Now, similarly, instead of "making x large enough," we will choose a number N and make x larger than N. Different values of M may require different values of N. For example, you can make x^2 larger than 4 by making x larger than 2, and you can make x^2 larger than 9 by making x larger than 3. The word "corresponding" in the following sentence emphasizes this fact.

For any number M, there is a corresponding number N such that you can make $f(x)$ larger than M by making x larger than N.

Finally, we rephrase our talk of "making things large" more mathematically:

For any number M, there is a corresponding number N such that if x is larger than N then $f(x)$ is larger than M.

Replacing "is larger than" with "$>$," and specifying that M and N are positive, gives Stewart's definition.

In this definition, "approaching ∞" is interpreted as "getting large," which is made more precise as "greater than a given number." Similarly, "approaching $-\infty$" is interpreted as "getting small," which is made more precise as "less than a given number." This is how we get the other three definitions from Section 2.

4 Polynomials of Degree 1

In this section, we consider the simple case in which $P(x)$ is a polynomial of degree 1. That is, $P(x) = ax + b$, where $a \neq 0$. We will show that the formulas of Section 1 hold in this case. Since in this case the degree of $P(x)$ is odd, we must show that

$$a > 0 \implies \lim_{x\to\infty} P(x) = \infty;$$

$$a > 0 \implies \lim_{x\to-\infty} P(x) = -\infty;$$

$$a < 0 \implies \lim_{x\to\infty} P(x) = -\infty;$$

$$a < 0 \implies \lim_{x\to-\infty} P(x) = \infty.$$

We will show that if $a < 0$ then $\lim_{x\to\infty} P(x) = -\infty$. The arguments in the other three cases are similar.

According to the definition, to show that

$$\lim_{x\to\infty} P(x) = -\infty,$$

we must show that for every number M, there is a (positive) number N such that

$$x > N \implies P(x) < -M.$$

To find N, we can start with M and work backwards:

$$P(x) < -M \iff ax + b < -M \iff ax < -M - b \iff x > -\frac{(M+b)}{a}.$$

(In the last step, we use the fact that $a < 0$, so multiplying by a reverses the sign of the inequality.) Therefore, we can choose N to be any (positive) number such that $N \geq -\frac{M+b}{a}$, and we will have that

$$x > N \implies P(x) < -M.$$

This is what we needed.

5 Proof by Induction

In this section, we discuss the technique of proof by induction, which will be used to prove the formulas of Section 1 hold for polynomials of any positive degree. The reader who is comfortable with proof by induction may skip this section.

We will suppose that $\varphi(n)$ is a statement expressing some property of a natural number n. For example, $\varphi(n)$ could be the statement

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

Then $\varphi(0)$ is the statement

$$\sum_{i=0}^{0} i = \frac{0(0+1)}{2}.$$

Evaluating each side, we get

$$0 = \frac{0}{2},$$

which is clearly true. Similarly, $\varphi(3)$ is the statement

$$\sum_{i=0}^{3} i = \frac{3(3+1)}{2}.$$

Evaluating each side, we get

$$0 + 1 + 2 + 3 = \frac{12}{2},$$

which is again true.

We can go on checking $\varphi(n)$ for particular values of n, but we can never check all possible cases. Proof by induction is a technique that is used to show that some statement $\varphi(n)$ is true for *every* natural number n, or possibly for *every* natural number $n \geq b$ (where b is some particular natural number).

The intuition behind proof by induction is as follows. Suppose that φ is true at 0 (that is, $\varphi(0)$ is true). Suppose also that whenever φ is true at some natural number, then it is also true at the next one (that is, $\varphi(n) \implies \varphi(n+1)$). Then φ must be true at every natural number. (It is true at 0; since it is true at 0, it is true at the next number, 1; since it is true at 1, it is true at the next number, 2; and so forth.)

Therefore, in order to prove that φ is true at every natural number n (or at every natural number $n \geq b$), it will suffice to prove two things:

1. φ is true at 0 (or at b);
2. if φ is true at some natural number then it is also true at the next one ($\varphi(n) \implies \varphi(n+1)$).

To prove $\varphi(n) \implies \varphi(n+1)$, the general technique is to assume $\varphi(n)$ and, using that assumption, prove $\varphi(n+1)$. This may be confusing at first; you are trying to prove that $\varphi(n)$ is true for every natural number n, so assuming $\varphi(n)$ may seem circular. However, it is not actually circular, because you are *not* assuming that $\varphi(n)$ is true for *every* natural number n. You are assuming that n is *some particular number* for which φ happens to be true, and showing that in that case, $n+1$ is another number for which φ is true.

The method of proof by induction, using some standard terminology, is this:

> To prove $\varphi(n)$ for every natural number n (or every natural number $n \geq b$) by induction on n, do two things:
>
> 1. Base Case: Prove $\varphi(0)$ (or prove $\varphi(b)$).
> 2. Inductive Step: Assume $\varphi(n)$. Use this to prove $\varphi(n+1)$.
>
> The assumption $\varphi(n)$ in the inductive step is called the *inductive hypothesis*. When you are proving $\varphi(n+1)$, instead of "because $\varphi(n)$ is true," you may say, "by inductive hypothesis."

Here is an example of a proof by induction.

Proposition 5.1. *For every natural number n,*

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

Proof. We prove this by induction on n.

Base Case: For $n = 0$ we must prove that

$$\sum_{i=0}^{0} i = \frac{0(0+1)}{2}.$$

Evaluating each side, we get

$$0 = 0,$$

which is clearly true.

Inductive Step: Assume that the proposition is true for n, that is,

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

This is the inductive hypothesis.

We must show the proposition is true for $n+1$, that is,

$$\sum_{i=0}^{n+1} i = \frac{(n+1)(n+2)}{2}. \tag{*}$$

The lefthand side of $(*)$ can be rewritten as

$$\sum_{i=0}^{n+1} i = \left(\sum_{i=0}^{n} i\right) + (n+1).$$

By inductive hypothesis, we rewrite this as

$$\left(\sum_{i=0}^{n} i\right) + (n+1) = \frac{n(n+1)}{2} + (n+1).$$

We rewrite this using algebra as

$$\frac{n(n+1)}{2} + (n+1) = \frac{n^2+n}{2} + \frac{2n+2}{2} = \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2}.$$

Since this equals the righthand side of $(*)$, we are done with the inductive step.

This completes the proof.

□

6 Polynomials of Any Degree

In this section we prove the formulas of Section 1 hold for polynomials of any positive degree.

Proposition 6.1. *For every natural number $n \geq 1$, if $P(x)$ is a polynomial of degree n with leading coefficient a, then*

$$\lim_{x\to\infty} P(x) = \begin{cases} \infty & \text{if } a > 0; \\ -\infty & \text{if } a < 0. \end{cases}$$

and

$$\lim_{x\to-\infty} P(x) = \begin{cases} \begin{cases} \infty & \text{if } a > 0, \\ -\infty & \text{if } a < 0, \end{cases} & \text{if } n \text{ is even;} \\ \\ \begin{cases} -\infty & \text{if } a > 0, \\ \infty & \text{if } a < 0, \end{cases} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We prove this by induction on n.

Base Case: For $n = 1$, the truth of the proposition was proven in Section 4.

Inductive Step: Assume that the proposition is true for polynomials of degree n. We must show the proposition is true for polynomials of degree $n + 1$.

Suppose, then, that $P(x)$ is a polynomial of degree $n + 1$ with leading coefficient a. We will show that the proposition is true for $P(x)$.

We will do this for the case where $n + 1$ is even and $a < 0$; the other three cases are similar. In this case, we will show that

$$\lim_{x\to-\infty} P(x) = -\infty;$$

again, the case of $\lim_{x\to\infty} P(x)$ is similar.

To show

$$\lim_{x\to-\infty} P(x) = -\infty,$$

we must show that for every positive number M, we can find a corresponding positive number N such that if $x < -N$, then $P(x) < -M$.

Let M be any positive number. We show how to find a corresponding N.

We can write

$$P(x) = ax^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0.$$

Setting

$$Q(x) = ax^{n-1} + b_{n-1}x^{n-2} + \cdots + b_1,$$

we have

$$P(x) = xQ(x) + b_0.$$

Therefore, we can write

$$P(x) < -M \iff xQ(x) + b_0 < -M \iff xQ(x) < -(M + b_0) \iff -xQ(x) > M + b_0.$$

We must find an N such that if $x < -N$, then $-xQ(x) > M + b_0$.

Since $Q(x)$ has degree n, we know by inductive hypothesis that the proposition is true for $Q(x)$. The degree of $Q(x)$ is n, which is odd, and the leading coefficient of $Q(x)$ is a, which is negative. Therefore,

$$\lim_{x \to -\infty} Q(x) = \infty.$$

The definition in Section 1, then, tells us that for every positive number $\overline{M}$, there is a positive number $\overline{N}$, such that if $x < -\overline{N}$, then $Q(x) > \overline{M}$.

Choose $\overline{M}$ to be any positive number such that $\overline{M} \geq M + b_0$, and let $\overline{N}$ be any positive number such that if $x < -\overline{N}$, then $Q(x) > \overline{M}$. Let N be any positive number such that $N \geq \overline{N}$ and $N \geq 1$. We must show that if $x < -N$, then $-xQ(x) > M + b_0$. Since $\overline{M} > M + b_0$, it will suffice to show that $-xQ(x) > \overline{M}$.

Suppose, then, that $x < -N$.

Since $N \geq \overline{N}$, we have $x < -N \leq -\overline{N}$. Therefore, $Q(x) > \overline{M}$. Since x is negative, $-x$ is positive, and so we can multiply by $-x$ to get $-xQ(x) > -x\overline{M}$. Now, since $x \leq -N$, we have $-x \geq N \geq 1$. Because $\overline{M}$ is positive and $-x \geq 1$, we have $-x\overline{M} \geq \overline{M}$. Putting this together with $-xQ(x) > -x\overline{M}$, we get $-xQ(x) > \overline{M}$, which is what we needed to show.

This completes the proof

□

References

[1] Stewart, James. *Calculus*, seventh edition. Cengage Learning, 2012.

Notes for Math 17 authors:

This document serves two purposes. First, it is an example of the format of a mathematics paper. Second, it is a source you may cite if you wish to use the facts about limits of polynomials proven here:

> Groszek, Marcia. *Limits of Polynomials at Infinity.* Classroom handout, Dartmouth College, Math 17, Winter 2015.

You may wonder how much detail you need to go into in your explanations. Think of your reader as another student in the class, who may not understand the material quite as well as you do.[1] For example, in this paper I discussed proof by induction, because beginning Math 17 students may never have seen this technique. However, since we will discuss proof by induction in class, you will be able to assume that your reader understands this technique.

Pictures and diagrams can be helpful sometimes. However, unless you have time on your hands and are up for a project, you don't need to learn how to draw them electronically. Hand-drawn pictures, provided they are clear and neat, are perfectly acceptable.

[1] Even if you think you are the worst student in the class, assume your reader does not understand things as well as you do. This will enhance the clarity of your prose.

The authorial "we" and the authorial "I" are both acceptable, although you should be consistent.

The citation style I used is standard for mathematics papers. I encourage you to use this style, but you may use any footnote or endnote style you are used to, as long as you are consistent and include a list of references. You may not use a style that places citations parenthetically in the text. Mathematics is hard enough to read without cluttering it up with things that could very well go elsewhere.

Shirley Zhao, the math librarian at Kresge Library, has created a research guide for our course, which you can find at this web address:

http://researchguides.dartmouth.edu/math17_hilbert10th

The resources under "Writing Help" can answer most of your questions about format and style. The most useful of the online resources is probably "Writing a Math Phase Two Paper." (Under "LaTeX help" you will find resources to help you with LaTeX, the typesetting program that I used to create this paper, and that you will use to create your papers. Ms. Zhao will come to our class on Friday, January 16, to help you get started with LaTeX.)

Excellent mathematical writing style embodies several characteristics, of which the three most important are clarity, clarity, and clarity. It is important to use words precisely and correctly. Generally, simple declarative sentences and consistent word use are preferable to variation in sentence structure and vocabulary. The same is true of most technical writing; the deeper and more complex the ideas, the simpler and more transparent the writing should be. My favorite quotation about this comes from the web page "Guidelines for Writing a Phiilosophy Paper" by NYU philosophy professor James Pryor:[2]

> If your paper sounds as if it were written for a third-grade audience, then you've probably achieved the right sort of clarity.

[2]http://www.jimpryor.net/teaching/guidelines/writing.html