Math 17
Winter 2015
Written Exercises Assigned January 7

These exercises are due at the beginning of class on Friday, January 9.

Recall that, from now on, we assume our unknowns denote natural numbers.

A *Diophantine expression* is an expression of the form

$$(\exists x_1)(\exists x_2)\cdots(\exists x_n)(D(a_1,\ldots,a_k,x_1,\ldots,x_n)=0),$$

where D is a polynomial with unknowns $a_1,\ldots,a_k,x_1,\ldots x_n$ and integer coefficients. (Recall that "$(\exists x)\cdots$" means "there exists a natural number x such that $\cdots$.")

A *Diophantine set* is a subset A of $\mathbb{N}^k$ that is defined by a Diophantine expression; that is, a set of the form

$$A=\{(a_1,\ldots,a_k)\mid(\exists x_1)(\exists x_2)\cdots(\exists x_n)(D(a_1,\ldots,a_k,x_1,\ldots,x_n)=0)\}.$$

We say the Diophantine expression

$$(\exists x_1)(\exists x_2)\cdots(\exists x_n)(D(a_1,\ldots,a_k,x_1,\ldots,x_n)=0)$$

defines the set A.

Example: Give a Diophantine expression defining

$$\{a\mid a\text{ is odd}\}.$$

Solution: We can rewrite "a is odd" as "$(\exists x)(a=2x+1)$," which we can then rewrite in the official form of a Diophantine expression:

$$\boxed{(\exists x)(a-(2x+1)=0)}.$$

(This is a solution to the problem, because it shows that

$$\{a\mid a\text{ is odd}\}=\{a\mid(\exists x)(a-(2x+1)=0)\}.)$$

For the following problems, you need only give an appropriate Diophantine expression. You do not need to show how you got it, as long as it is correct. However, if it is not correct, showing how you got it can net you some partial credit.

For these problems, show the set is Diophantine by giving a Diophantine expression defining the set.

1. $\{a \mid 3 \text{ is the remainder when } a \text{ is divided by } 4\}$.

2. $\{(a, b) \mid b \text{ is the remainder when } a \text{ is divided by } 4\}$.

   Notice that this isn't just a matter of replacing 3 in the previous problem with b, because when you divide a number by 4, the remainder is less than 4.