Math 217
Winter 2015
Tuesday, January 13

These written homework exercises are due on Friday, January 16.

1. Do exercise (1) below.
2. Explain why exercise (3) below tells us that the set of natural numbers that can be written as the sum of four squares is closed under multiplication.
3. Do exercise (4) below. (You may use exercise (3) and the fact that every natural number greater than 1 can be expressed as the product of primes.)

Exercises from Monday's class, with a partial solution or two:

1. Suppose that $p = 2n + 1$ is a prime number.

   (a) Let
   $$X = \{0^2, 1^2, \ldots, n^2\}.$$
   Show that if a and b are any two distinct numbers in X, then $a \not\equiv b \mod p$.

   (b) Let
   $$Y = \{-1 - (0^2), -1 - (1^2), \ldots, -1 - (n^2)\} = \{-1 - x \mid x \in X\}.$$
   Show that if c and d are any two distinct numbers in Y, then $c \not\equiv d \mod p$.

   (c) Explain why there must be numbers $a \in X$ and $c \in Y$ such that $a \equiv c \mod p$.

   (d) Conclude that some number of the form Np, with $0 < N < p$, can be written as the sum of three squares.

2. Show by example that if $q = 2n + 1$ is not prime, and
   $$X = \{0^2, 1^2, \ldots, n^2\},$$
   there may be two distinct numbers in X that are congruent modulo q.

   Solution: If $n = 3$, then $q = 2n + 1 = 9$ is not prime. The set X in this case is
   $$\{0^2, 1^2, 2^2, 3^2, 4^2\} = \{0, 1, 4, 9, 16\},$$
   which contains two elements (0 and 9) that are congruent modulo 9.

   A different solution: If $n = 7$, then $q = 2n + 1 = 15$ is not prime. In this case we have $1^2 \equiv 4^2 \mod 15$.

   I included this solution to give an example in which two nonzero elements of X are congruent. Notice, here we have $4^2 - 1^2 = (4 + 1)(4 - 1) = 5 \cdot 3$. You can see just how the proof you gave in exercise (1) fails to work in the case that $2n + 1$ is not prime.

3. Verify the Euler identity:

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2) \cdot (y_1^2 + y_2^2 + y_3^2 + y_4^2) =$$
$$(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 +$$
$$(x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3)^2 +$$
$$(x_1y_3 - x_3y_1 - x_2y_4 + x_4y_2)^2 +$$
$$(x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2)^2$$

(Don't try multiplying out both sides and canceling every term in sight. It would work, but... Use symmetry, and the fact that every term you get by multiplying out the right hand side of the equation has one of two forms, either $x_i^2y_j^2$, or else $\pm x_ix_ky_jy_\ell$ where $i \neq k$ and $j \neq \ell$ (and either i, j, k, ℓ are all different, or $\{i, k\} = \{j, \ell\}$).)

Partial Solution: First, it would work perfectly well to multiply out both sides and see that you get the same thing, it's just a lot of algebraic manipulation. (That is, it would work providing you can do all that algebraic manipulation without errors.)

We can instead look at the terms on each side, and classify them into cases.

When we multiply out the lefthand side, we get the sum of all possible terms of the form $x_i^2y_j^2$.

The righthand side is the sum of four expressions $(a+b+c+d)^2$, where a, b, c, and d represent terms of the form x_ij_j. When we multiply out these expressions, the terms we get include the a^2, b^2, c^2, and d^2; from the four expressions $(a+b+c+d)^2$, these terms give us every possible term of the form $x_i^2y_j^2$. That is, they give us all the terms on the lefthand side.

We also get some other terms on the righthand side, terms from $(a+b+c+d)^2$ such as ab. We can see that these terms are of one of two forms, $x_iy_jx_jy_i$ or $x_iy_jx_ky_\ell$. We need to show that each of these terms on the righthand side is canceled out by another term of the same sort on the righthand side.

In the case $x_iy_jx_jy_i$, this term can be written as $(x_iy_i)(x_jy_j)$ or as $(x_iy_j)(x_jy_i)$. The term $(x_iy_i)(x_jy_j)$ comes from multiplying out the first of the squares on the righthand side, and occurs with a plus sign. The term $(x_iy_j)(x_jy_i)$ comes from multiplying out one of the other squares on the righthand side, and occurs with a minus sign. (We can check that in every case, x_iy_j and x_jy_i occur with opposite signs). Therefore, the two occurrences of $x_iy_jx_jy_i$ cancel each other out.

You can apply the same kind of reasoning to the case of terms of the form $x_iy_jx_ky_\ell$, to see that each occurs on the righthand side once with a plus sign and once with a minus sign.

4. We will show that every odd prime can be written as the sum of four squares. Show it follows from this that every natural number can be written as the sum of four squares.