Math 17
Winter 2015
Written Exercises due Monday, February 9

1. Show that if $b \geq 2^n + 1$ then $b > \binom{n}{k}$ for every $k \leq n$. (This means, as we showed in class, that if $b \geq 2^n + 1$ then the triple $((b+1)^n, b, n+1)$ codes the sequence

$$\left\langle \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n} \right\rangle.$$

See also Chapter 3, Section 4 in the textbook. In class, we used $b > n^n$ instead, as it is easier to verify that this is big enough.)

Hint: Write 2^n as $(1+1)^n$, and expand it out using the binomial theorem,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

2. Design a Turing machine, which we will call MULTNOW, that will do the following:

Suppose the contents of the tape contain a sequence $0111\cdots1110$ having n-many 1's, and end with a sequence $0111\cdots111$ having k-many 1's. (These are two different sequences, although $n = k$ is possible. It is also possible that $n = 0$, $k = 0$, or both.) When started with the tape head on the leftmost 0 of the first sequence, MULTNOW ends with the final sequence of k-many 1's replaced with a sequence of nk-many 1's, and the contents of the tape otherwise the same as when the machine was started.

The tape may contain other symbols than $*, 0, 1, 2, 3, \lambda$. You may assume that $*$ occurs only in the leftmost cell, that empty cells (or cells containing λ) occur only to the right of all cells containing symbols (other than λ), and that the symbols 2, 3, and 4 occur nowhere on the tape.

Note that the Turing machines we have seen so far use only the symbols $*, 0, 1, 2, 3, \lambda$. You may use them as building blocks, if you explain how to modify them to work properly (properly for your purposes) if there are other symbols around. Also feel free to use the symbol 4.

Note also, however, that the multiplication machines we have seen so far are MULT(i, j), a different machine for each (i, j), so they can't very well be used as building blocks.

3. Design a Turing machine, which we will call CHECK, that will do the following:

When started with a tape whose contents include exactly two cells containing the symbol 5, and exactly two cells containing the symbol 6, with both 6's to the right of both 5's, CHECK will halt in state YES if the number of 1's between the two 5's is equal to the number of 1's between the two 6's and in state NO otherwise. The machine should halt with the contents of the tape the same as when it started.

As in the previous problem, there may be other symbols of various sorts on the tape. In particular, there may be symbols other than 1's between the 5's and between the 6's, which CHECK should not count. You may assume that $*$ occurs only in the leftmost cell, that empty cells (or cells containing λ) occur only to the right of all cells containing symbols (other than λ), and that the symbols 2, 3, and 4 occur nowhere on the tape.

4. (Bonus question for extra credit.) In class, we showed that if n is large enough relative to m, then

$$m! = n^m \, div \binom{n}{m}$$

where $div(a, b)$ is the integer part of $\frac{a}{b}$. This proof, minus a few details, is also in Chapter 3, Section 4 of the textbook.

In class, we claimed, but did not prove, that $n \geq (m+1)^{m+2}$ is sufficiently big. Verify this claim.