Dartmouth College

Mathematics 17

Assignment 6

due Wednesday, February 15

1. Using the fundamental theorem for finite abelian groups, list a complete set of nonisomorphic abelian groups of order $7^3 11^4$.

2. For $n \geq 1$, U_n is a finite abelian group, For $5 \leq n \leq 15$, use your knowledge of these groups to characterize them as in the fundamental theorem. For example, U_3 is a group of order 2, a prime, so U_3 is a cyclic group of order 2, that is $U_3 \cong \mathbb{Z}_2$. The group U_8 is an abelian group of order 4, so by the fundamental theorem isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or to $\mathbb{Z}_4$. We easily check for all $a \in U_8$ that $a^2 = 1$, so U_8 is not cyclic, and so $U_8 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

   This next series of questions concerns finding points of intersection of curves in affine and projective plane ($\mathbb{A}^2(F)$ or $\mathbb{P}^2(F)$ where $F = \mathbb{R}$ or $\mathbb{C}$). Recall the affine plane $\mathbb{A}^2(\mathbb{R})$ is just a fancy way to say $\mathbb{R}^2$. Points in the affine plane have coordinates (x, y) while points in the projective plane have coordinates $[x, y, z]$, and we identify points $(x, y) \in \mathbb{A}^2$ with the point $[x, y, 1] \in \mathbb{P}^2$. The points of $\mathbb{P}^2$ not coming from $\mathbb{A}^2$ have the form $[x, y, 0]$ where x, y are not both zero.

3. Consider the parabola $y - x^2 = 0$ and the line $x = 2$.
   (a) Find their point(s) of intersection in $\mathbb{A}^2(\mathbb{R})$.
   (b) Now we "homogenize" the affine curves and consider the projective curves $yz - x^2 = 0$ and $x - 2z = 0$. Note that setting $z = 1$ produces (dehomogenizes) the projective curves producing the affine ones. Find the points of intersection in $\mathbb{P}^2(\mathbb{R})$. You should at least recover the affine points.

4. Consider the affine curves $x + y + 2 = 0$ and $x^2 + y^2 = 1$.
   (a) Find all points of intersection in $\mathbb{A}^2(\mathbb{R})$, in $\mathbb{A}^2(\mathbb{C})$.
   (b) Now consider their projective counterparts: $x + y + 2z = 0$ and $x^2 + y^2 = z^2$. Determine any extra solutions that appear in $\mathbb{P}^2(\mathbb{R})$ or $\mathbb{P}^2(\mathbb{C})$ that did not come from affine points, or explain why there are none.

5. The vertical lines $x = 2$ and $x = 3$ do not intersect in the affine plane. Find the point of intersection of their projective counterparts: $x = 2z$, $x = 3z$.

6. Find the point of intersection of the projective lines $y = mx + bz$ and $y = mx + b'z$ if $b \neq b'$.