Basics of matrices

As before, see (for example) Linear Algebra and its Applications *by David Lay for a more thorough (and better) introduction.*

A matrix is an $m \times n$ array of numbers, e.g.

$$\underset{3\times 2}{\begin{bmatrix} 2 & 0 \\ 3 & -1 \\ 4 & 2 \end{bmatrix}}, \quad \underset{4\times 1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \quad \underset{3\times 5}{\begin{bmatrix} -1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & x_1 & 0 & 0 & 1 \end{bmatrix}}$$

The transpose of a matrix A, denoted A^T, is obtained by swapping the rows and columns, e.g.

$$\begin{bmatrix} 2 & 0 \\ 3 & -1 \\ 4 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \end{bmatrix}$$

To multiply matrices $A \cdot B$, we require the number of columns in the left matrix to be the number of rows in the right matrix. Then, for example, we have, for a row vector $\bar{a} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$ and column vector $\bar{b} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{bmatrix}$

$$\bar{a} \cdot \bar{b} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \cdot \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{bmatrix} = [a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41}]$$

And for a general matrix, where $\bar{a}_i$ and $\bar{b}_j$ are the rows of A and columns of B respectively, we have

$$\underset{m\times n}{A \cdot B} = \underset{m \times p}{\begin{bmatrix} \leftarrow & \bar{a}_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \bar{a}_m & \rightarrow \end{bmatrix}} \cdot \underset{p \times n}{\begin{bmatrix} \uparrow & & \uparrow \\ \bar{b}_1 & \cdots & \bar{b}_n \\ \downarrow & & \downarrow \end{bmatrix}} = [c_{ij}]$$

where $c_{ij} = \bar{a}_i \cdot \bar{b}_j = a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1p}b_{p1}$

Notice that $A \cdot B \neq B \cdot A$, indeed these might even have different sizes.

The $n \times n$ identity matrix I_n has 1's on the leading diagonal and 0's elsewhere.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}$$

It is such that (whenever the multiplication is defined), $A \cdot I_n = A$ and $I_n \cdot B = B$.

Gaussian elimination

This is analogous to the method for solving systems of linear equations. The allowed operations are:

- *Multiply a row (column) by a nonzero constant*
- *Add a multiple of a row (column) to another row (column)*
- *Swap two rows (columns)*

To try to invert a matrix A, we form the augmented matrix $[A|I]$, where I is an identity matrix of the same size as A. We apply Gaussian elimination to the rows of this to try to reduce A to I. If we succeed, we reach some $[I|B]$, then $A = B^{-1}$. If we fail, we will reduce a row of A to contain only 0's, and conclude here that A is not invertible.

For example, if $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ then $[A|I] = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$, and applying row operations yields $\begin{bmatrix} 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & -1 \end{bmatrix}$, so $A^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0 & -1 \end{bmatrix}$.

This works because an invertible matrix is a product of elementary matrices, left-multiplication by which correspond to the operations of Gaussian elimination, e.g. $A = E_1 \cdot E_2 \cdot \ldots \cdot E_k$. Each operation is invertible, so by applying the appropriate sequence of operations we get

$$E_k^{-1} \cdot \ldots \cdot E_2^{-1} \cdot E_1^{-1} \cdot A = E_k^{-1} \cdot \ldots \cdot E_2^{-1} \cdot E_1^{-1} \cdot E_1 \cdot E_2 \cdot \ldots \cdot E_k = I$$

Applying the same operations to I, we get

$$E_k^{-1} \cdot \ldots \cdot E_2^{-1} \cdot E_1^{-1} \cdot I = E_k^{-1} \cdot \ldots \cdot E_2^{-1} \cdot E_1^{-1}$$

so we see that $B = E_k^{-1} \cdot \ldots \cdot E_2^{-1} \cdot E_1^{-1} = A^{-1}$.