Chap. 1

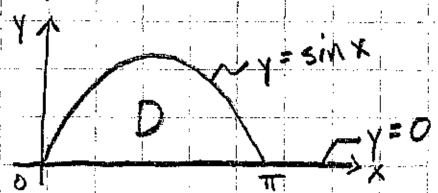
1 d) $\rho(x) = \frac{2x}{x^2+1} \quad a = 0, \; b = 1$

$$Q = \int_a^b \rho(x)\,dx = \int_0^1 \frac{2x}{x^2+1}\,dx \qquad \text{let } u = x^2+1 \quad du = 2x\,dx$$

$$\Longrightarrow Q = \int_{x=0}^{1} \frac{du}{u} = \ln u \Big|_{x=0}^{1} = \ln(x^2+1)\Big|_0^1 = \ln 2$$

For a, b, c, & e; set up the integral in the same way and evaluate using the bag o' tricks you developed in high school.

2 a) $\rho(x,y) = y$

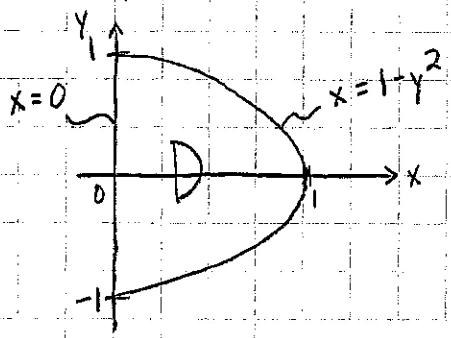


$$\iint_D \rho\,dA = \int_0^{\pi}\int_0^{\sin x} y\,dy\,dx$$

$$= \int_0^{\pi}\left(\frac{1}{2}y^2\Big|_0^{\sin x}\right)dx = \int_0^{\pi} \frac{1}{2}\sin^2 x\,dx = \frac{\pi}{4}$$

(from calc., book, or integration by parts)

d) $\rho(x,y) = 2x$

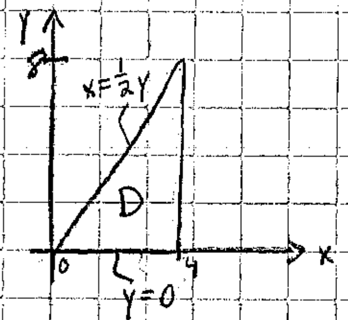


$$\iint_D \rho\,dA = \int_{-1}^{1}\int_0^{1-y^2} 2x\,dx\,dy$$

$$= \int_{-1}^{1}\left(x^2\Big|_0^{1-y^2}\right)dy = \int_{-1}^{1}(1-y^2)^2\,dy = \frac{16}{15}$$

For b & c set up the same way, but remember to pick the order of integration that's easiest.

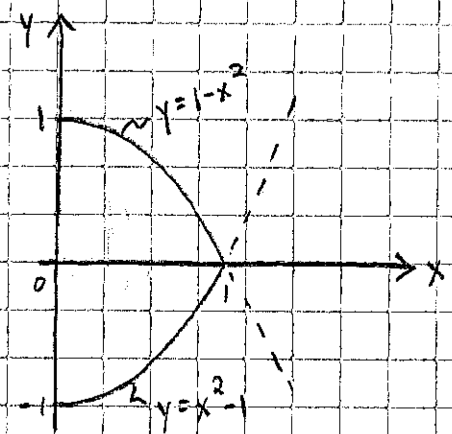
3 a) $\int_0^4 \int_0^{2x} \rho \, dy \, dx \qquad 0 \le y \le 2x \quad 0 \le x \le 4$



Reversing order of integration:

$$\int_0^8 \int_{\frac{1}{2}y}^4 \rho \, dx \, dy$$

b) $\int_0^1 \int_{x^2-1}^{1-x^2} \rho \, dy \, dx \qquad x^2 - 1 \le y \le 1 - x^2 \quad 0 \le x \le 1$



Need to break into two integrals, one for $-1 \le y \le 0$ & one for $0 \le y \le 1$.

For $-1 \le y \le 0$, we have $0 \le x \le \sqrt{1+y}$

For $0 \le y \le 1$, we have $0 \le x \le \sqrt{1-y}$

Reversing order of integration:

$$\int_{-1}^0 \int_0^{\sqrt{1+y}} \rho \, dx \, dy + \int_0^1 \int_0^{\sqrt{1-y}} \rho \, dx \, dy$$

4 $\rho(x, y) = x + y$

$$\text{total mass} = \int_0^1 \int_0^1 (x+y) \, dx \, dy = \int_0^1 \left(\frac{1}{2}x^2 + yx \Big|_0^1 \right) dy = \int_0^1 \left(\frac{1}{2} + y \right) dy = 1$$

$$C_x = \frac{1}{\text{total mass}} \iint_D x \rho(x,y) \, dA = \int_0^1 \int_0^1 (x^2 + xy) \, dx \, dy$$

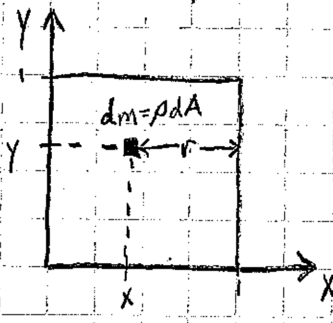
$$= \int_0^1 \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 y \Big|_0^1 \right) dy = \int_0^1 \left(\frac{1}{3} + \frac{1}{2}y \right) dy = \frac{1}{3}y + \frac{1}{4}y^2 \Big|_0^1 = \frac{7}{12}$$

By symmetry, $C_y = \frac{7}{12}$

$\Longrightarrow C_M = \left(\frac{7}{12}, \frac{7}{12} \right)$

$$I_{x\text{-axis}} = \iint_D y^2 \rho \, dA = \int_0^1\int_0^1 (xy^2 + y^3)\,dx\,dy = \frac{5}{12}$$

For moment of inertia about $x=1$ & $y=x$, need to derive the integral. (I'll only do $x=1$)



For a point particle, $I = \frac{1}{2} m r^2$, so for our tiny chunk, we have

$$dI = \frac{1}{2}\,dm\,r^2 = \frac{1}{2}\rho(1-x)^2\,dA$$

$$\Rightarrow I = \iint_D \frac{1}{2}(1-x)^2 \rho \, dA$$ (I'll let you actually compute the thing)

5 $$\iiint_D z\,dV = \int_{-1}^{1}\int_0^{1-x^2}\int_{-\sqrt{1-x^2-z}}^{\sqrt{1-x^2-z}} z\,dy\,dz\,dx$$

6 a) $-1 \le y \le 1 \quad x^2+z^2 \le 1$

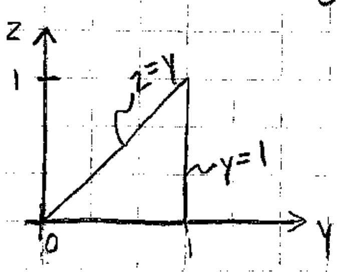
$$\iiint_D \rho\,dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \rho\,dx\,dz\,dy$$

b) $x^2+y^2+z^2 \le 1$

$$\iiint_D \rho\,dV = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} \rho\,dx\,dy\,dz$$

7 $0 \le z \le y \quad 0 \le y \le 1 \quad 0 \le x \le 1$

A slice at a given x looks like:



$$\int_0^1\int_0^1\int_0^y dz\,dy\,dx = \int_0^1\int_0^1\int_z^1 dy\,dz\,dx$$

8 In cylindrical coordinates: $x^2+y^2+z^2 = r^2+z^2 \le 1$

$$Q = \iiint_D \rho\, dV = \frac{150}{4\pi}\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} z^2\, r\, dz\, dr\, d\theta = \frac{150}{4\pi}\int_0^{2\pi}\int_0^1 \frac{2}{3}(1-r^2)^{3/2} r\, dr\, d\theta$$

$$= 50\int_0^1 (1-r^2)^{3/2} r\, dr \qquad \text{let } u = 1-r^2 \quad du = -2r$$

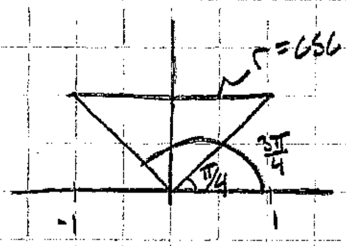
$$\Rightarrow Q = -25\int_{u=1}^{0} u^{3/2}\, du = -25\left(\frac{2}{5}u^{5/2}\Big|_1^0\right) = 10$$

9 1.

$0 \le y \le \sqrt{1-x^2}$
$0 \le x \le 1$

$$\int_0^1\int_0^{\sqrt{1-x^2}} \rho\, dy\, dx = \int_0^{\frac{\pi}{2}}\int_0^1 \rho\, r\, dr\, d\theta$$

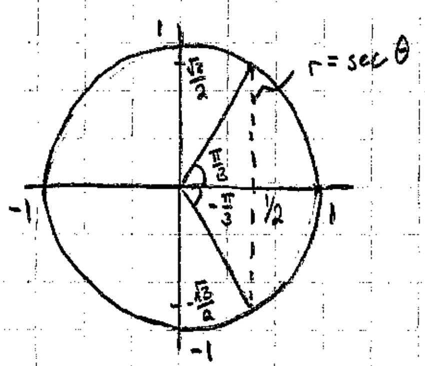
3.



$0 \le r \le \csc\theta$
$\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\csc\theta} \rho\, r\, dr\, d\theta = \int_{-1}^1\int_{|x|}^1 \rho\, dy\, dx$$

10 $$Q_{tot} = \int_0^{2\pi}\int_0^1 r^3\, dr\, d\theta = \frac{\pi}{2}$$



$$Q_{slice} = \frac{\pi}{2} - 2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{\sec\theta}^{1} r^3\, dr\, d\theta$$

$$= \frac{\pi}{2} - 2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left(\frac{1}{4} - \frac{1}{4}\sec^4\theta\right) d\theta$$

$$= \frac{\pi}{2} - \frac{\pi}{3} + \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^4\theta\, d\theta$$

Plug in to Maple (or double integration by parts).

11 $0 \le z \le 1 - x^2 - y^2 = 1 - r^2$

$$\int_0^{2\pi}\int_0^1\int_0^{1-r^2} z\, r\, dz\, dr\, d\theta = 2\pi \int_0^1 \left(\frac{1}{2} z^2 r \Big|_0^{1-r^2} \right) dr = \pi \int_0^1 (1-r^2)^2 r\, dr = \frac{\pi}{6}$$

12 $E_z = \int_0^{2\pi}\int_0^R \frac{Br}{4\pi\varepsilon_0} \frac{zr}{(r^2+z^2)^{3/2}}\, dr\, d\theta$ Plug in to Maple (or double integration by parts).

13 I did this one in cylindrical coordinates earlier (if you want to see it in Cartesian, then you are a masochist).