LECTURE OUTLINE

Polar Derivatives and Curve Geometry

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Math 15

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Goals

Polar Derivatives

Curvature

Accelration’s Components

For the Record

A *parameterized curve* for us is a mapping of the interval (a, b) into space given by $(x(\tau), y(\tau), z(\tau))$ for $\tau \in (a, b)$; where $x(\tau)$, $y(\tau)$, $y(\tau)$ are asssumed to be differentiable function of the *parameter* τ.

A *nice curve* parameterized by $\vec{r}(\tau)$ will mean a parameterized curve where $\frac{d\vec{r}}{d\tau}(\tau) \neq 0$ and $\vec{r}(\tau_0) \neq \vec{r}(\tau_1)$ when τ_0 and τ_1 are distinct numbers in (a, b).

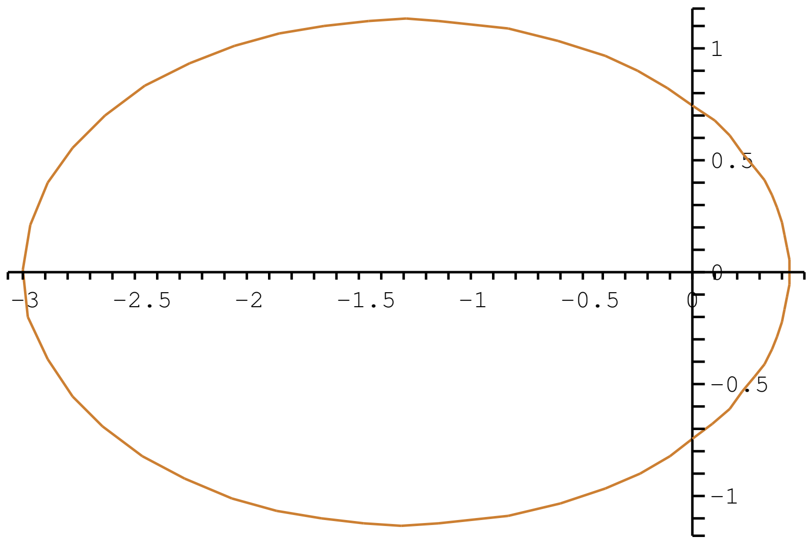
Ellipse Review

We found that the point traveling along the ellipse described by $(\frac{ed}{1+e\cos(t)}, t)_P$ satisfied

$$\frac{d\vec{r}}{dt} = \frac{ed\sin(t)}{(1+e\cos(t))^2}\hat{i} + \frac{ed(e+\cos(t))}{(1+e\cos(t))^2}\hat{j},$$

and using $\vec{w} = (\vec{w}\cdot\hat{e}_1)\hat{e}_1 + (\vec{w}\cdot\hat{e}_2)\hat{e}_2 + (\vec{w}\cdot\hat{e}_3)\hat{e}_3$ we found

$$\frac{d\vec{r}}{dt} = \frac{e^2d\sin(t)}{(1+e\cos(t))^2}\hat{r} + \frac{ed}{1+e\cos(t)}\hat{\theta}.$$


1
0.5
0
-0.5
-1
-3
-2.5
-2
-1.5
-1
-0.5
0

Derivatives in Polar Coordinates

$$
\begin{aligned}
\frac{d\hat{r}}{dt} &= \dot{\theta}\hat{\theta} \\
\frac{d\hat{\theta}}{dt} &= -\dot{\theta}\hat{r} \\
\frac{d\vec{r}}{dt} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\
\frac{d^2\vec{r}}{dt^2} &= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} \\
&= (\ddot{r} + Centrifugal)\hat{r} + (Coriolis + r\ddot{\theta})\hat{\theta}
\end{aligned}
$$

Ellipse Part 2

Take our point following the path $(\frac{ed}{1+e\cos(t)}, t)_P$, and using the formulas on the previous slide find this point's velocity and acceleration in polar coordinates.

Our Speedometer

Imagine that we take a nice curve parameterized by $\vec{r}(t)$ for t in $[0, b]$, and lay it gently along the s-axis. (Assume we lay it down on the positve s direction with $\vec{r}(0)$ placed at $s = 0$). We know that $\vec{r}(t)$ is sent to $s(t) = \int_0^t |\frac{d\vec{r}}{dt}|dt$. This $s(t)$ motion along the s axis contains our "speedometer's speed and acceleration". (Imaigne we are in a car and can only look at our watch and the speedometer, but not out the window of a windy road).

Geometry of a Curve

We have

$$\frac{d\vec{r}}{dt} = \frac{ds}{dt}\frac{d\vec{r}}{ds} \equiv \frac{ds}{dt}\hat{T}$$

and

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2s}{dt^2}\hat{T} + (\frac{ds}{dt})^2\frac{d\hat{T}}{ds} \equiv \frac{d^2s}{dt^2}\hat{T} + (\frac{ds}{dt})^2(k\hat{N})$$

where $k(s) = |\frac{d\vec{T}}{ds}|$ is called the curves *curvature* at $\vec{r}(s)$, and $\hat{T} \cdot \hat{N} = 0$.

Ellipse Part 3

For each t, find the curvature of our ellipse $(\frac{ed}{1+e\cos(t)}, t)_P$.