AEX $\boxed{22}$ Determine whether each series $\sum_{n=1}^{\infty} a_n$ is convergent, and if it is, find the sum.

1) $a_n = \dfrac{2 \cdot 3^n}{2^{2n}}$

First, we rewrite a_n: $a_n = \dfrac{2 \cdot 3^n}{2^{2n}} = \dfrac{2 \cdot 3^n}{(2^2)^n}$

$$= 2 \cdot \frac{3^n}{4^n} = 2 \cdot \left(\frac{3}{4}\right)^n.$$

Thus $\sum a_n$ is geometric with 1^{st} term $= 2 \cdot \left(\frac{3}{4}\right)^1$ & ratio $\frac{3}{4}$. (since $\frac{3}{4} < 1$, the $\sum$ converges!)

Hence $\displaystyle\sum_{n=1}^{\infty} a_n = \frac{1^{st} \text{ term}}{1 - \text{ratio}} = \frac{\frac{6}{4}}{1 - \frac{3}{4}} = \frac{6}{4-3} = 6$

2) $a_n = \left(\dfrac{1}{3^n}\right)^{-1}$. rewrite: $a_n = \left(\frac{1}{3^n}\right)^{-1} = (3^{-n})^{-1} = 3^n$.

Geometric with ratio $= 3 > 1$, so the sum diverges.

3) $a_n = \dfrac{n+1}{n}$. Well, $\displaystyle\lim_{n \to \infty} \frac{n+1}{n} = 1 \neq 0$, so the sum diverges.

4) $a_n = (-1)^n$. $\displaystyle\lim_{n \to \infty} (-1)^n$ doesn't exist! So it certainly doesn't go to 0. $\Rightarrow$ the sum diverges.

AEX 23 In each case, for which values of x does the ratio test guarantee convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$

1) $a_n = n$. $b_n = a_n x^n$

ratio test: $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot x\right|$

can pull out $|x|$ because it is constant w.r.t. n.

$= |x| \cdot \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = x \cdot 1 = x$. For the ratio test to guarantee convergence, $|x| < 1$.

So the values of x are in $\{x : |x| < 1\}$.

2) $a_n = \frac{1}{n!}$. $b_n = a_n x^n = \frac{x^n}{n!}$

ratio test: $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{x^{n+1}}{(n+1)!}\right)}{\left(\frac{x^n}{n!}\right)}\right| = \lim_{n\to\infty}\left|\frac{x \cdot n!}{(n+1)!}\right|$

$= \lim_{n\to\infty}\left|\frac{x \cdot \cancel{n!}}{(n+1)\cancel{n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0$. Thus no matter what x is the ratio test guarantees convergence. $\Rightarrow x \in \{y : y \in \mathbb{R}\} = \mathbb{R}$.

AEX 23] 3) $a_n = 2^n$. $b_n = 2^n x^n$.
ctd

ratio test: $\lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{2^n x^n} \right| = \lim_{n\to\infty} |2 \cdot x|$

$= |2x|$. If less than 1, r.t. gives convergence

$|2x| < 1 \iff |x| < \frac{1}{2}$. So $x \in \{y : |y| < \frac{1}{2}\}$

$(= \{y : -\frac{1}{2} < y < \frac{1}{2}\})$

4) $a_n = n^n$, $b_n = n^n x^n$

It should be obvious that if $x \neq 0$, the sum will diverge, so we will show this with the ratio test: (Suppose $x \neq 0$...)

$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1} x^{n+1}}{n^n x^n} \right| \geq \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)^n} \frac{x^{n+1}}{x^n} \right|$

$= \lim_{n\to\infty} |(n+1)x| = \infty$, since $x \neq 0$.

Thus the ratio test gives us that the sum converges <u>only</u> when $x = 0$.

AEX 24 Find the Taylor series for $f(x)$ around $x=a$, and then find its radius of convergence!

1) $f(x) = \ln(x)$; $a=1$.

From previous homework,
$f^{(0)}(x) = \ln(x)$, $f^{(0)}(1) = 0$, & for $k \geq 1$,

$f^{(k)}(1) = (-1)^{k-1}(k-1)!$ Thus the Taylor series for f around $x=a$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = \cancel{\frac{f^{(0)}(1)(x-1)^0}{0!}} + \sum_{n=1}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!(x-1)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cancel{(n-1)!}(x-1)^n}{n \cdot \cancel{(n-1)!}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}.$$

To get the radius of convergence, we use the ratio test:

$$\lim_{n\to\infty} \left| \frac{\left(\frac{(-1)^{n+1-1}(x-1)^{n+1}}{n+1}\right)}{\left(\frac{(-1)^{n-1}(x-1)^n}{n}\right)} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot (x-1) \right| = |x-1|.$$

$|x-1| < 1 \Leftrightarrow -1 < x-1 < 1 \Leftrightarrow 0 < x < 2$. Diameter $= 2$, so radius $= 1$.

AEX 24 2) $f(x) = \frac{1}{2x+1}$; $a=0$.

We could use Taylor's formula for this, but it's easier to just recognize that this is the result of summing a geometric series:

$$f(x) = \frac{1}{2x+1} = \frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2x)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n 2^n x^n. \Leftarrow \text{ Taylor series around } x=a=0.$$

Again, we could use the ratio test, but we don't have to this time.

Geometric series converge $\Leftrightarrow$ |ratio| < 1.

$\Leftrightarrow |(-2x)| < 1 \Leftrightarrow 2|x| < 1 \Leftrightarrow |x| < \frac{1}{2}$

$\Leftrightarrow -\frac{1}{2} < x < \frac{1}{2}$. Diameter $= 1$, so radius $= \frac{1}{2}$.

AEX $\boxed{24}$ 3) $f(x)=x^3$, $a=-1$.

cont'd

$f^{(0)}(x)=x^3$	$f^{(0)}(-1)=(-1)^3=-1$
$f^{(1)}(x)=3x^2$	$f^{(1)}(-1)=3(-1)^2=3$
$f^{(2)}(x)=6x$	$f^{(2)}(-1)=6(-1)=-6$
$f^{(3)}(x)=6$	$f^{(3)}(-1)=6$
$f^{(4)}(x)=0$	$f^{(4)}(-1)=0$
$\vdots$	$\vdots$
$f^{(k)}(x)=0$	

So Taylor Series $=\sum_{n=0}^{\infty}\frac{f^{(n)}(-1)(x-(-1))^n}{n!}=\sum_{n=0}^{\infty}\frac{f^{(n)}(-1)(x+1)^n}{n!}$

$$=\frac{(-1)(x+1)^0}{0!}+\frac{3\cdot(x+1)^1}{1!}+\frac{-6\cdot(x+1)^2}{2!}+\frac{6\cdot(x+1)^3}{3!}+0+0+\cdots$$

$$=(-1)+3(x+1)-\frac{6}{2}(x+1)^2+\frac{\cancel{6}(x+1)^3}{\cancel{6}}$$

$$=-1+3x+3-3(x^2+2x+1)+(x^3+3x^2+3x+1)$$

$$=\cancel{-1}+\cancel{3x}+\cancel{3}-\cancel{3x^2}-\cancel{6x}-\cancel{3}+x^3+\cancel{3x^2}+\cancel{3x}+\cancel{1}$$

$=x^3$. $\leftarrow$ this is a finite sum, so the series always converges, regardless of x.

Thus Diameter $=\infty$, & radius "$=\frac{\infty}{2}$" $=\infty$.

$\boxed{AEX\ 24}$ 4) $f(x) = e^x$; $a = 2$

ctd

$f^{(0)}(x) = e^x \Rightarrow$ Taylor series $= \boxed{\sum_{n=0}^{\infty} \frac{e^2 \cdot (x-2)^n}{n!}}$

$f^{(1)}(x) = e^x$

$\vdots \qquad = e^x$

$\qquad\quad \vdots$

ratio test: $\lim_{n\to\infty} \left| \frac{\left(\frac{\cancel{e^2} \cdot (x-2)^{n+1}}{(n+1)!}\right)}{\left(\frac{\cancel{e^2} \cdot (x-2)^n}{n!}\right)} \right| = \lim_{n\to\infty} \left| \frac{\cancel{n!}}{(n+1)\cancel{n!}} (x-2) \right|$

$= \lim_{n\to\infty} \left| \frac{x-2}{n+1} \right| = 0$, regardless of x.

$\Rightarrow$ Diameter $= \infty \Rightarrow$ radius $= \infty$.

AEX 25 (a) Find the Taylor series for $f(x)=e^x$, $f(x)=\cos(x)$, and $f(x)=\sin(x)$, around the point $x=0$.

$$f(x)=e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\left(=:P_{exp}(x)\right)$$

$\cos^{(0)}(x)=\cos(x)$	1	$\sin^{(0)}(x)=\sin(x)$	0
$\cos^{(1)}(x)=-\sin(x)$	0	$\sin^{(1)}(x)=\cos(x)$	1
$\cos^{(2)}(x)=-\cos(x)$	-1	$\sin^{(2)}(x)=-\sin(x)$	0
$\cos^{(3)}(x)=\sin(x)$	0	$\sin^{(3)}(x)=-\cos(x)$	-1
$\cos^{(4)}(x)=\cos(x)$	1	$\sin^{(4)}(x)=\sin(x)$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

So $\cos(x)=\sum_{K=0}^{\infty}\frac{(-1)^K\cdot x^{2K}}{(2K)!}\left(=:P_{\cos}(x)\right)$ $\quad$ $\sin(x)=\sum_{K=0}^{\infty}\frac{(-1)^K\cdot x^{2K+1}}{(2K+1)!}\left(=:P_{\sin(x)}\right)$

$$\text{(b) } P_{exp}(i\theta)=\sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}=\sum_{n=0}^{\infty}\frac{i^n\theta^n}{n!}\overset{\text{rearranging terms}}{=}\sum_{n=0}^{\infty}\frac{i^{2n}\theta^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\frac{i^{2n+1}\theta^{2n+1}}{(2n+1)!}$$

$$=\sum_{n=0}^{\infty}\frac{(i^2)^n\theta^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\frac{(i^2)^n(i)\theta^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n}}{(2n)!}+i\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}$$

$$=P_{\cos}(\theta)+i\cdot P_{\sin}(\theta).$$

(c) ✓