1. Use the power series method to solve the following initial value problems. Assume $c \neq 0$.

(a) $\frac{d^2 f}{dt^2} = c^2 f$ with $f(0) = 1$ and $\frac{df}{dt}(0) = 0$.

Steps

① Assume $f(t) = \sum_{n=0}^{\infty} b_n t^n$ & find the b_n.

② Plug in $\sum_{n=2}^{\infty} n(n-1) b_n t^{n-2} = c^2 \sum_{n=0}^{\infty} b_n t^n$

or rather $\sum_{n=2}^{\infty} n(n-1) b_n t^{n-2} - \sum_{n=2}^{\infty} c^2 b_{n-2} t^{n-2} = 0$

or rather $\sum_{n=2}^{\infty} \left(n(n-1) b_n - c^2 b_{n-2}\right) t^{n-2} = 0$

③ so $b_n = \frac{c^2 b_{n-2}}{n(n-1)}$ (*) for each $n \geq 2$

④ for $f(0) = 1 = b_0$ & $\frac{df}{dt}(0) = 0 = b_1$

$b_2 = \frac{c^2 b_0}{2 \cdot 1} = \frac{c^2}{2 \cdot 1}$ $b_3 = \frac{c^2 \cdot 0}{3 \cdot 2} = 0$

$\vdots$ $\vdots$

$b_{2n} = \frac{c^{2n}}{(2n)!}$ $b_{2n+1} = 0$

(See proof on next page)

⑤ finally $\boxed{f_a(t) = \sum_{n=0}^{\infty} \frac{c^{2n}}{(2n)!} t^{2n}}$ is the need power series.

(b) $\frac{d^2 f}{dt^2} = c^2 f$ with $f(0) = 0$ and $\frac{df}{dt}(0) = c$.

well from step 3 a. we know

$$b_n = \frac{c^2 b_{n-2}}{n(n-1)}$$

step 4 b.

$f(0) = 0 = b_0$ $\qquad$ $\frac{df}{dt}(0) = c = b_1$

$b_2 = \frac{c^2 \cdot b_0}{2 \cdot 1} = \frac{c^2 \cdot 0}{2 \cdot 1} = 0$ $\qquad$ $b_3 = \frac{c^2 b_1}{3 \cdot 2 \cdot 1} = \frac{c^3}{3 \cdot 2 \cdot 1}$

$\vdots$ $\qquad$ $\vdots$

$b_{2n} = 0$ $\qquad$ $b_{2n+1} = \frac{c^{2n+1}}{3 \cdot 2 \cdot 1}$

5 b. $\boxed{f_b(t) = \sum_{n=0}^{\infty} \frac{c^{2n+1}}{(2n+1)!} t^{2n+1}}$ is the needed power series

Not need for this class, but...

proof: part 4 a. Note $b_0 = 1 = \frac{c^{2 \cdot 0}}{0!}$, $b_1 = 0$

Assume $b_{2(n-1)} = \frac{c^{2(n-1)}}{(2(n-1))!}$, Now $b_{2n} = \frac{c^2 b_{2(n-1)}}{2(n-1)!} = \frac{c^2 c^{2(n-1)}}{(2n)(2n-1)(2n-2)!}$

$= \frac{c^{2n}}{(2n)!}$ as need

So by induction $b_{2n} = \frac{c^{2n}}{(2n)!}$

Similiar, for b_{2n+1} & for 4 b.

2. (a) Find the radius of convergence of the power series constructed in problems 1a and 1b.

Steps

(1) Recoll the ratio test

if $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = r < 1$ then $\sum_{n=0}^{\infty} b_n$ converges

$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = r > 1$ then $\sum_{n=0}^{\infty} b_n$ diverges

(2) 1a well $\lim_{n\to\infty}\left|\frac{\frac{c^{2(n+1)}t^{2(n+1)}}{(2(n+1))!}}{\frac{c^{2n}t^{2n}}{(2n)!}}\right| = \lim_{n\to\infty}\frac{(ct)^2}{(2(n+1))(2n+1)}$

$= \frac{(ct)^2}{4}\lim_{n\to\infty}\frac{1}{n^2+\frac{3}{2}n+\frac{1}{2}}$

$= 0 < 1$

(3) so by (1) & (2) $\sum_{n=0}^{\infty}\frac{c^{2n}x^{2n}}{(2n)!}$ converges for all t

& radius of convergence is $\boxed{\infty}$

Similary, for 1b since $\lim_{n\to\infty}\left|\frac{\frac{(ct)^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(ct)^{2n+1}}{(2n+1)!}}\right| =$

$= \lim_{n\to\infty}\frac{(ct)^2}{(2(n+1)+1)(2(n+1))} = 0$ & (3) still holds.

(b) Show the derivative of the function constructed in problem 1a is a constant multiple of the function in constructed in problem 1b.

well

$$\frac{d}{dt}\left(f_a(t)\right) = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{c^{2n}t^{2n}}{(2n)!}$$

$$= \sum_{n=1}^{\infty}\frac{2n\,c^{2n}t^{2n-1}}{(2n)!}$$

$$= \sum_{n=1}^{\infty}\frac{c^{2n}\,t^{2n-1}}{(2n-1)!}$$

$$= \sum_{n=0}^{\infty}\frac{c^{2(n+1)}\,t^{2(n+1)-1}}{(2(n+1)-1)!}$$

$$= c\sum_{n=0}^{\infty}\frac{c^{2n+1}t^{2n+1}}{(2n+1)!} = \boxed{c\left(f_b(t)\right)}$$

3. Recall $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$, and let $\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$ and $sech(z) = \frac{1}{\cosh(z)}$.

(a) Using the above formula, show that $\cosh(x)^2 - \sinh(x)^2 = 1$ and $\tanh(x)^2 + sech(x)^2 = 1$.

$$(\cosh(x))^2 - (\sinh(x))^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

foil

$$= \left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - \left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right)$$

$$= \frac{2 - (-2)}{4} = \frac{4}{4} = 1$$

$$\& \quad \frac{1}{(\cosh(x))^2}\left[(\cosh(x))^2 - (\sinh(x))^2 = 1\right]$$

or rather

$$\left(\frac{\cosh(x)}{\cosh(x)}\right)^2 - \left(\frac{\sinh(x)}{\cosh x}\right)^2 = \frac{1}{(\cosh(x))^2}$$

or rather $1 - (\tanh(x))^2 = (sech(x))^2$

so $(\tanh(x))^2 + (sech(x))^2 = 1$ as needed.

(b) Show that $\frac{d}{dx}\sinh(x) = \cosh(x)$ and that $\frac{d}{dx}\cosh(x) = \sinh(x)$.

$$\frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\parallel$$

$$\frac{d}{dx}\left(\sinh(x)\right)$$

$$\& \quad \frac{d}{dx}\left(\cosh(x)\right) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{e^x + (-e^{-x})}{2} = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

(c) Find a power series expression for $\sinh(z)$ and $\cosh(z)$.

$$\boxed{\sinh(z)} = \frac{e^{z} - e^{-z}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\right)$$

split into even & odd parts

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} - \left(\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\right)\right)$$

$$= \frac{1}{2}\left(2 \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} + 0\right) = \boxed{\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}}$$

$$\boxed{\cosh(z)} = \frac{e^{z} + e^{-z}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\right)$$

$$= \frac{1}{2}\left(2 \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\right) = \boxed{\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}}$$

(d) Let $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ and $sech(z) = \frac{1}{\cosh(x)}$. Show $\frac{d}{dx}\tanh(x) = 1 - \tanh(x)^2$ and $\frac{d}{dx}sech(x) = -sech(x)\tanh(x)$.

$$\frac{d}{dx}\left(\tanh(x)\right) = \frac{d}{dx}\left(\left(\sinh(x)\right)\left(\cosh(x)\right)^{-1}\right)$$

$\boxed{\text{from (c)}}$

$$= \cosh(x)\cdot\left(\cosh(x)\right)^{-1} + \sinh(x)(-1)\left(\cosh(x)\right)^{-2}\cdot\sinh(x)$$

$$= 1 - \left(\frac{\sinh(x)}{\cosh(x)}\right)^2 = 1 - \left(\tanh(x)\right)^2$$

$$\frac{d}{dx}\left(sech(x)\right) = \frac{d}{dx}\left(\cosh(x)\right)^{-1}$$

$$= -\left(\cosh(x)\right)^{-2}\cdot\sinh(x)$$

$$= -\frac{\sinh(x)}{\cosh(x)}\cdot\frac{1}{\cosh(x)}$$

$$= -\tanh(x)\cdot sech(x)$$

or Use ... cosh & with 3(c)

4. (a) Show that $\sinh(ct)$ and $\cosh(ct)$ are solutions the the differential equation $\frac{d^2f}{dt^2} - c^2 f = 0$ when $c \neq 0$. (Hint you may use problems 1 and 3).

(b) Solve the initial value problem $\frac{d^2f}{dt^2} - c^2 f = 0$ with $f(0) = A$ and $\frac{df}{dt}(0) = B$ using $\sinh(ct)$ and $\cosh(ct)$.

4(a) well $\frac{d^2}{dt^2}(\sinh(ct)) = c\frac{d}{dt}(\cosh(ct))$ (by 3b.)

$= c^2 \sinh(ct)$ (by 3b)

& similarly $\frac{d^2}{dt^2}\cosh(ct) = c\frac{d}{dt}(\sinh(ct)) = c^2\cosh(ct)$

4(b) we know that any thing in the form $f(t)$ $c_0 \sinh(ct) + d_0 \cosh(t)$ is a solution.

At $t=0$

$A = f(0) = \left(c_0 \sinh(ct) + d_0 \cosh(ct)\right)\Big|_{t=0} = c_0\left(\frac{1-1}{2}\right) + d_0 \frac{1+1}{2} = d_0$

&

$B = \frac{df}{dt}(0) = \left(c c_0 \cosh(ct) + c d_0 \sinh(ct)\right)\Big|_{t=0}$

$= c c_0 \frac{1+1}{2} + c d_0 \left(\frac{1-1}{2}\right) = c c_0$

So

$\boxed{f(t) = \frac{B}{c}\sinh(ct) + A\cosh(ct)}$ is the solution.

Steps

5. Find a general solution to $\frac{d^2f}{dt^2} - c^2 f = e^{2t}$. (Hint you may use problem 4).

① Well if we can find a particular solution $f_p(t)$ then any solution is in the form $f_p(t) + f_h(t)$ where $f_h(t)$ solves $\frac{d^2 f_h}{dt^2} - c^2 f_h = 0$. From ④ we know $f_h(t) = \frac{B}{c}\sinh(ct) + A\cosh(ct)$ we only need an $f_p(t)$.

② the Guess method well we guess $f_p(t) = a_0 e^{2t}$ & note this implies $a_0\left(4e^{2t} - c^2 e^{2t}\right) = e^{2t}$ or $a_0(4-c^2) = 1$. So if $\boxed{c \neq \pm 2}$

$$\boxed{f(t) = \frac{1}{4-c^2} e^{2t} + \frac{B}{c}\sinh(ct) + A\cosh(ct)}$$

if $\boxed{c = \pm 2}$ we guess $a_0 t e^{2t}$ & find

$$a_0\left(\frac{d}{dt}\left(e^{2t} + 2te^{2t}\right) - c^2 t e^{2t}\right) = e^{2t}$$

$$a_0\left((4-c^2)te^{2t} + (4)e^{2t}\right) = e^{2t} \quad \text{or } a_0 = \frac{1}{4}$$

& $f(t) = \frac{t}{4}e^{2t} + \frac{B}{c}\sinh(ct) + A\cosh(ct)$

6. Recall in polar coordinates that $\hat{r}(\theta)$ parameterizes the unit circle and that $\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$, a $\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$. (To solve this problem you may use the results of problem 3.)

(a). Recall the equation of a hyperbola is given by $x^2 - y^2 = c$ for $c \neq 0$. Explain why $\hat{r}_h(\theta_h) = \cosh(\theta_h)\hat{i} + \sinh(\theta_h)\hat{j}$ parameterizes a hyperbola.

a piece of a hyperbola

Well by 3(a)

$$(\cosh(\theta_h))^2 - (\sinh(\theta_h))^2 = 1,$$

as needed.

(b). Let $\hat{\theta}_h = \sinh(\theta_h)\hat{\imath} + \cosh(\theta_h)\hat{\jmath}$. Show $\frac{d\hat{r}_h}{dt} = \dot{\theta}_h\hat{\theta}_h$.

well

$$\frac{d\hat{r}_h}{dt} = \frac{d}{dt}\left(\left(\cosh(\theta_h)\right)\hat{i} + \left(\sinh(\theta_h)\right)\hat{j}\right)$$

$$= \frac{d}{dt}\left(\cosh(\theta_h)\right)\hat{i} + \frac{d}{dt}\left(\sinh(\theta_h)\right)\hat{j}$$

$$\text{by } 3b \; = \; \left(\sinh(\theta_h)\right)\dot{\theta}_h\,\hat{i} + \sinh(\theta_h)\,\dot{\theta}_h\,\hat{j}$$

$$= \dot{\theta}_h\,\hat{\theta}_h$$

(c). Can $\frac{d\hat{\theta}_h}{dt}$ be expressed as a multiple of $\hat{r}_h$? If so derive a formula relating the two, if not explain why not.

Well

$$\frac{d\hat{\theta}_h}{dt} = \frac{d}{dt}\left(\sinh(\theta_h)\,\hat{i} + \cosh(\theta_h)\,\hat{j}\right)$$

$$= \left(\frac{d}{dt}\sinh(\theta_h)\right)\hat{i} + \left(\frac{d}{dt}\cosh(\theta_h)\right)\hat{j}$$

$$= \cosh(\theta_h)\,\dot{\theta}_h\,\hat{j} + \sinh(\theta_h)\,\dot{\theta}_h\,\hat{j}$$

$$= \dot{\theta}_h\,\hat{r}_h$$

So indeed!

(d). Why do the hats in $\hat{r}_h$ and $\hat{\theta}_h$ feel inappropriate?

Well

$$|\hat{r}_h| = (\cosh(x))^2 + (\sinh(x))^2$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh(2x) \neq 1$$

except for very special x.

not needed

In fact $\frac{e^{2x} + e^{-2x}}{2} = 1$ when

$$e^{4x} - 2e^{2x} + 1 = 0$$

so $e^{2x} = \frac{2 \pm \sqrt{4-4}}{2} = \pm 1$

so $x = 0$ is the only such value

7. How is the area of the parallelogram determined by $\hat{r}_h(\theta_h)$ and $\hat{\theta}_h(\theta_h)$ from problem 6 changing as θ_h changes? (Hint: use problem 3.)

Recall the area of this parallelogram is

$$\left| \hat{r}_h(\theta_h) \times \hat{\theta}_h(\theta_h) \right|$$

$$\| $$

$$\left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cosh(\theta_h) & \sinh(\theta_h) & 0 \\ \sinh(\theta_h) & \cosh \theta_h & 0 \end{vmatrix} \right| = \left| (\cosh(\theta_h))^2 - (\sinh(\theta_h))^2 \right|$$

$= 1$ by (3a) So this are $\boxed{\text{does not change}}$ as θ_h varies.

8. (a) Let $\vec{F} = y^2\hat{i} + x\hat{j}$ and suppose an object travels along $\gamma(t) = (t - \tanh(t), sech(t))$. Set up (but do not evaluate!) an integral for the work done by this force on this object as time goes from 0 to 2.

$$W_{ork} = \int_\gamma \vec{F} \cdot dr$$

$$= \int_0^2 \vec{F}(t - \tanh(t), \operatorname{sech}(t)) \cdot \frac{d}{dt}(t - \tanh(t), \operatorname{sech}(t))\, dt$$

$$= \int_0^2 \left((\operatorname{sech}(t))^2, t - \tanh(t)\right) \cdot \left(1 - (1 - (\tanh(t))^2), -\operatorname{sech}(t)\tanh(t)\right) dt$$

(hx 3d)

$$\boxed{= \int_0^2 (\operatorname{sech}(t)\tanh(t))^2 - \operatorname{sech}(t)\tanh(t)\,(t - \tanh(t))\, dt}$$

(b) Let $\vec{F} = y^2\hat{\imath} + 2xy\hat{\jmath}$ and suppose an object travels along $\gamma(t) = (t - \tanh(t), sech(t))$. Set up (but do not evaluate!) an integral for the work done by this force on this object as time goes from 0 to 2.

Just as in 6d.

$$\boxed{\text{work} = \int_0^2 (\mathrm{sech}(t)\tanh(t))^2 - 2(\mathrm{sech}(t))^2 \tanh(t)\,(t - \tanh(t))\,dt}$$

steps

9. Compute the work done in either problem 8a or problem 8b. (Hint: be sure to think).

① We these integrals stink!

We better find one of the $\vec{F}$ is conservative

Well $-\frac{\partial V}{\partial x} = y^2 \qquad V = -xy^2 + f(y)$

& $-\frac{\partial V}{\partial y} = 2xy \qquad V = -xy^2 + g(x)$

which is consistant so $V(x,y) = -xy^2$

is our need potential.

② So Work $= \int_\gamma \vec{F}\cdot d\vec{r}$ for any γ $(0,1) \to \left(2-\left(\frac{e^2-e^{-2}}{e^2+e^{-2}}\right), \frac{2}{e^2+e^{-2}}\right) = (x_0, y_0)$

or rather

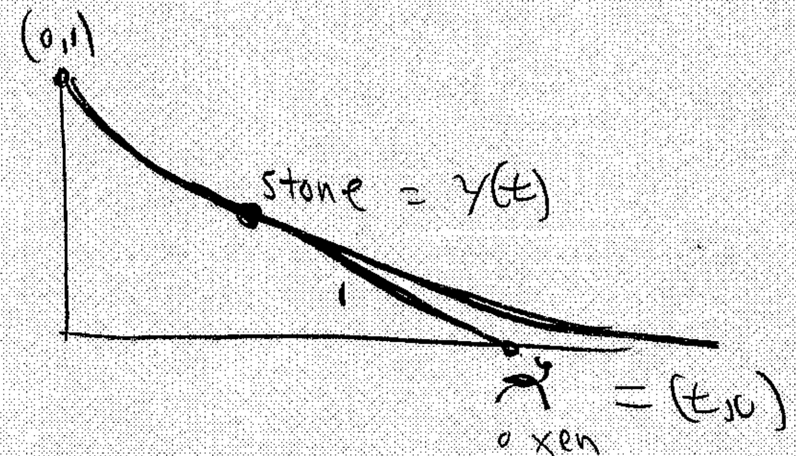
$$-\int_\gamma \vec{F}\cdot dr = V(x_0,y_0) - V(0,1)$$

$$= -x_0 y_0^2 + 2\cdot 0\cdot 1$$ So

$$\text{Work} = \boxed{+\left(2-\left(\frac{e^2-e^{-2}}{e^2+e^{-2}}\right)\right)\frac{4}{(e^2+e^{-2})^2}}$$

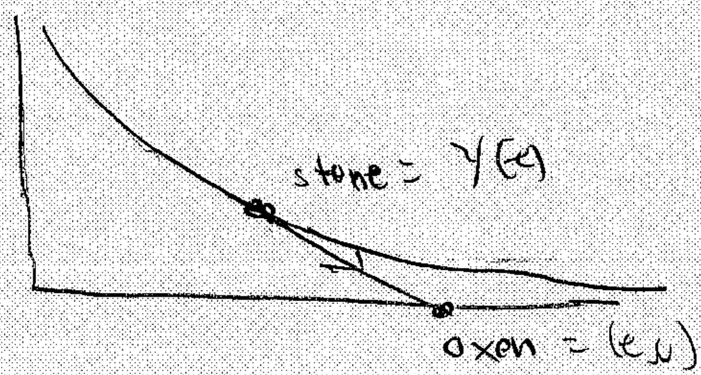
10. Imagine an oxen staring at $(0,0)$ is attached via a taught rope of length one decameter to large stone at $(0,1)$. Suppose the oxen is then driven at a constant speed of one decameter per minute along the $x-axis$ Let $\gamma(t)$ denote the position of the stone at time t. (To solve this problem you may use the results of problem 3.)

(a). Justify in words and/or a picture why the path must satisfy that the line segment starting at $\gamma(t)$ and ending at $(t,0)$ must have length 1.



The line segment from $\gamma(t)$ to $(t,0)$ is our taught rope of length 1.

(b). Justify in words and/or a picture why the line starting at $\gamma(t)$ heading in the direction determined by $\frac{d\gamma}{dt}$ must hit $x - axis$ at $(t, 0)$.



Well $\frac{d\gamma}{dt}$ describes the stone's instantaneous rate & direction of change; which is towards to oxen, since the oxen is pulling the stone. The oxen is a (t,0) on the x-axis, as needed.

(c). Show that the path $\gamma(t) = (t - \tanh(t), sech(t))$ satisfies the conditions described in 10a and 10b.

(10a) $$\left| (t - \tanh(t), \operatorname{sech}(t)) - (t, 0) \right| = \left| \frac{d\gamma}{dt} - (t, 0) \right|$$

(*)

$$= \left| (-\tanh(t), \operatorname{sech}(t)) \right|$$

$$= (\tanh(t))^2 + (\operatorname{sech}(t))^2$$

$$= 1 \quad \text{by } 3d$$

Similarly

(10 b) $$\frac{d\gamma}{dt} \overset{\text{by}}{\underset{3d}{=}} \left(1 - (1 - (\tanh(t))^2), -\operatorname{sech}(t)\tanh(t)\right)$$

$$= \tanh(t) \left(\tanh(t), -\operatorname{sech}(t) \right)$$

which has the same direction as $\frac{d\gamma}{dt} - (t, 0)$ from (*), as needed

(Hard)

(EXTRA CREDIT 1) Explain why $\gamma(t) = (t - \tanh(t), sech(t))$ is the **only** path that can satisfy 10a and 10b.

Steps

① 10(a) implies $(x-t)^2 + y^2 = 1$ or $x = -\sqrt{1-y^2} + t$

& hence $\dot{x} = \frac{-y}{\pm\sqrt{1-y^2}}\dot{y} + 1$ (*)

Negative since t is increasing

② 10(b) implies $(\dot{x}, \dot{y}) = c(x-t, y)$

or $\frac{\dot{x}}{x-t} = c = \frac{\dot{y}}{y}$ & hence

$\dot{y} = \frac{y}{x-t}\dot{x} = \frac{-y}{\sqrt{1-y^2}}\dot{x}$ (**)

③ So

$\dot{y} \overset{\text{by (*) \& (**)}}{=} \frac{-y^2}{1-y^2}\dot{y} \pm \frac{y}{\sqrt{1-y^2}}$

rearranging

$\frac{\dot{y}}{1-y^2} = \left(1 + \frac{y^2}{1-y^2}\right)\dot{y} = \frac{-y}{\sqrt{1-y^2}}$ & finally

$\boxed{\dot{y} = -y\sqrt{1-y^2}}$.

④ $\frac{dt}{dy} = \frac{\pm 1}{y\sqrt{1-y^2}}$ has some unique solution s.t. $y=1$ when $t=0$, &

$y(t) = sech(t)$ satisfies

$\frac{dy}{dt} = \frac{d\,sech(t)}{dt} \overset{(3d)}{=} -sech(t)\tanh(t) \overset{(3d)}{=} -y\sqrt{1-y^2}$, as needed.

(d). The path $\gamma(t)$ describes our stones path. Set up (but do not evaluate) an integral to determine how far the stone has traveled in the first t_0 minutes.

$$\int_0^{t_0} \left|\frac{d\gamma}{dt}\right| dt \underset{\text{by } 3}{=} \int_0^{t_0} \left|\left((\tanh(t))^2, -\operatorname{sech}(t)\tanh(t)\right)\right| dt$$

$$= \int_0^{t_0} (\tanh(t))^4 + (\tanh(t))^2 (\operatorname{sech}(t))^2 \, dt$$

$$= \int_0^{t_0} (\tanh(t))^2 \left((\tanh(t))^2 + (\operatorname{sech}(t))^2\right) dt$$

$$\underset{\text{by } 3}{=} \int_0^{t_0} (\tanh(t))^2 \, dt$$

incidentally

$$= \ln(\cosh(t)) \Big|_0^{t_0} = \ln(\cosh(t_0))$$

Easier

(EXTRA CREDIT 2) Do you expect the stone will have traveled exactly as far, less far, or further than the oxen, and why?

Well

$$\left|\frac{dy}{dt}\right| = (\tanh(t))^2 = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 < 1$$

$$\left(\text{since} \quad e^x - e^{-x} < e^x < e^x + e^{-x}\right).$$

$$\text{so} \quad \int_0^{t_0} \left|\frac{dy}{dt}\right| dt < \int_0^{t_0} 1 \, dt = t_0$$

= distance traveled by oxen.