Math 14
Winter 2009
Monday, January 26

Special Homework III

This week's special homework is about the formula for arc length; the same ideas apply to the formula for a path integral in general.

Suppose γ is a smooth curve parametrized by the function $\vec{r}(t)$ for $a \leq t \leq b$, and $\dfrac{d\vec{r}}{dt}$ is continuous on an open interval containing the closed interval $[a, b]$.

We said that if we take a small enough time interval Δt, the portion of the curve γ between $\vec{r}(t)$ and $\vec{r}(t + \Delta)$ should be almost straight. Therefore, we said, we should be able to approximate the arc length of γ in the following way: For some large n, divide the time interval $[a.b]$ into n-many equal subintervals of length $\Delta t = \dfrac{b-a}{n}$, with the endpoints of the subdivision being $t_0 = a$, $t_1 = a + \Delta t$, $t_2 = a + 2\Delta t$, $\ldots$, $t_n = a + n\Delta t = b$. Then we approximate the arc length of the portion of the curve between $\vec{r}(t_{i-1})$ and $\vec{r}(t_i)$ by the straight line distance between those two points,

$$(arclength)_i \approx |\vec{r}(t_i) - \vec{r}(t_{i-1})| = |\vec{r}(t_{i-1} + \Delta t) - \vec{r}(t_{i-1})|,$$

and approximate the entire arc length of γ by

$$S(n) = \sum_{i=1}^{n} |\vec{r}(t_i) - \vec{r}(t_{i-1})|.$$

Finally, we said the arc length of γ should be

$$\lim_{n \to \infty} S(n).$$

We further said that, if $\dfrac{d\vec{r}}{dt}$ represents velocity, and therefore $\left|\dfrac{d\vec{r}}{dt}\right|$ represents speed, then over a small interval from t to $t + \Delta t$, the direction of travel and the speed should not change very much, and so the distance traveled should be more or less the speed at time t times the elapsed time Δt, or

$\left|\dfrac{d\vec{r}}{dt}(t)\right| \Delta t$. Therefore, we said that, if we divide the time interval $[a.b]$ into n-many equal subintervals as before, we should have, for each i,

$$|\vec{r}(t_i) - \vec{r}(t_{i-1})| = |\vec{r}(t_{i-1} + \Delta t) - \vec{r}(t_{i-1})| \approx (arclength)_i \approx \left|\frac{d\vec{r}}{dt}(t_{i-1})\right| \Delta t,$$

and we then concluded that we should have

$$R(n) = \sum_{i=1}^{n} \left|\frac{d\vec{r}}{dt}(t_{i-1})\right| \Delta t \approx \sum_{i=1}^{n} |\vec{r}(t_i) - \vec{r}(t_{i-1})| = S(n),$$

and finally that

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty} R(n) = \int_a^b \left|\frac{d\vec{r}}{dt}(t)\right| dt.$$

There is a gap in this reasoning. It is not hard to see that, for a fixed t, as n gets larger and larger, the difference between $|\vec{r}(t + \Delta t) - \vec{r}(t)|$ and $\left|\dfrac{d\vec{r}}{dt}(t)\right| \Delta t$ gets smaller and smaller, because $\dfrac{d\vec{r}}{dt}(t)$ is just the limit as $\Delta t \to 0$ of $\dfrac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$. However, even if the difference between the terms we are adding up gets smaller and smaller, the number of terms we are adding up to get $S(n)$ and $R(n)$ is getting larger and larger, so it is not obvious that the sums are getting closer and closer together.

To fill this gap we need one additional fact. Recall that by

$$\frac{d\vec{r}}{dt}(t) = \lim_{\Delta t\to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t},$$

we mean that, for every t, for every $\epsilon > 0$, there is a $\delta > 0$ such that, whenever we have $0 < |\Delta t| < \delta$, then

$$\left|\frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} - \frac{d\vec{r}}{dt}(t)\right| < \epsilon.$$

The value of ϵ may depend on the values of δ and t. However, given that $\dfrac{d\vec{r}}{dt}$ is continuous on an open interval containing the closed interval $[a, b]$, we can assume that we can choose ϵ in such a way that its value depends only on the value of δ and NOT on the value of t:

(**) For every $\delta > 0$, there is an $\epsilon > 0$ such that, for every $t \in [a, b]$, whenever we have $0 < |\Delta t| < \delta$, then

$$\left| \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} - \frac{d\vec{r}}{dt}(t) \right| < \epsilon.$$

Assignment: Prove the following propositions:

Proposition 1: Under the given assumptions, including (**), for every $\epsilon > 0$, there is an N large enough so that, for every $n > N$ and every i between 1 and n, we have

$$\left| |\vec{r}(t_{i-1} + \Delta t) - \vec{r}(t_{i-1})| - \left| \frac{d\vec{r}}{dt}(t_{i-1}) \right| \Delta t \right| < \epsilon \Delta t.$$

Hint for Proposition 1: First argue that, in general, if $|\vec{b} - \vec{a}| < \epsilon$ then $\big| |\vec{b}| - |\vec{a}| \big| < \epsilon$.

Proposition 2: Under the given assumptions, for every $\epsilon^* > 0$ there is an N^* large enough so that, for every $n > N^*$,

$$|S(n) - R(n)| < \epsilon^*.$$

Hint for Proposition 2: Apply Proposition 1 with $\epsilon = \dfrac{\epsilon^*}{b - a}$.