Math 14
Winter 2009
Monday, January 5

Special Homework II

We defined a function $F : \mathbb{R}^m \to \mathbb{R}^n$ to be *differentiable* at the point $\vec{a} \in \mathbb{R}_n$ if there is an affine function

$$A(\vec{x}) = T(\vec{x} - \vec{a}) + \vec{b}$$

whose graph is tangent to the graph of F at $\vec{x} = \vec{a}$. Here T is an $n \times m$ matrix with constant entries, $(\vec{x} - \vec{a})$ is written as a column vector with m entries, and $\vec{b}$ is a column vector with n entries.

If this is the case, we set $F'(\vec{a}) = T$.

We defined the graphs of A and F to be *tangent* at $\vec{x} = \vec{a}$ provided:

1. $A(\vec{a}) = F(\vec{a})$;
2. $\displaystyle\lim_{\vec{x} \to \vec{a}} \frac{F(\vec{x}) - A(\vec{x})}{|\vec{x} - \vec{a}|} = \vec{0}$.

Clearly if the graph of $A(\vec{x}) = T(\vec{x} - \vec{a}) + \vec{b}$ is tangent to the graph of F at $\vec{x} = \vec{a}$, then by condition (1) we must have $\vec{b} = \vec{F}(\vec{a})$. Hence we have

$$A(\vec{x}) = T(\vec{x} - \vec{a}) + \vec{b} = F'(\vec{a})(\vec{x} - \vec{a}) + F(\vec{a}).$$

Assignment: Suppose that the graphs of $F(x, y) = \langle F_1(x, y), F_2(x, y), F_3(x, y) \rangle$ and

$$A(x, y) = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \\ t_{31} & t_{32} \end{pmatrix} \begin{pmatrix} x - a_1 \\ x - a_2 \end{pmatrix} + \begin{pmatrix} F_1(a_1, a_2) \\ F_2(a_1, a_2) \\ F_3(a_1, a_2) \end{pmatrix}$$

are tangent at the point $(x, y) = (a_2, a_2)$. Show that

$$t_{21} = \frac{\partial F_2}{\partial x}(a_1, a_2) = \lim_{x \to a_1} \frac{F_2(x, a_2) - F_2(a_1, a_2)}{x - a_1}.$$

Use the ε-δ definition of limit.

Hint: Try this and then come to tutorial or office hours for help or hints. This is a tricky problem. It is fine to have someone else show you how to do it before you write it up.

Note: This works for any t_{ij} and any $F : \mathbb{R}^m \to \mathbb{R}^n$. What we see from this is:

If the graph of
$$A(\vec{x}) = T(\vec{x} - \vec{a}) + \vec{b}$$
is tangent to the graph of F at $\vec{x} = \vec{a}$, then the matrix $T = F'(\vec{a})$ must be the matrix of partial derivatives of F evaluated at the point $\vec{a}$:

$$F'(\vec{a}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\vec{a}) & \frac{\partial F_1}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial F_1}{\partial x_m}(\vec{a}) \\ \frac{\partial F_2}{\partial x_1}(\vec{a}) & \frac{\partial F_2}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial F_2}{\partial x_m}(\vec{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(\vec{a}) & \frac{\partial F_n}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial F_n}{\partial x_m}(\vec{a}) \end{pmatrix}.$$

Therefore, there is only one possible choice for the tangent approximation to $F(\vec{x})$ near the point $\vec{x} = \vec{a}$.

In practice, to check whether a function $F(\vec{x})$ is differentiable at $\vec{a}$ we do the following:

1. Compute the partial derivatives of F. If any of them are undefined at $\vec{x} = \vec{a}$, then F is not differentiable at $\vec{a}$.

2. If all the partial derivatives of F are defined in some open ball around $\vec{a}$ and *continuous* at $\vec{a}$, then F is differentiable at $\vec{a}$.

3. If neither (1) nor (2) answers the question, then we use limits to check whether the graph of the potential tangent approximation to F near $\vec{a}$ is actually tangent. Because we know the partial derivatives of F at $\vec{a}$, we know what this tangent approximation would be.

Actually, there may be other things we can do if (1) and (2) don't work. For example, we know that if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$, then we can compute the directional derivatives of f at $\vec{a}$ using the gradient ∇f.

Therefore, if we can show that there is some unit vector $\vec{u}$ for which the directional derivative

$$\frac{\partial f}{\partial \vec{u}}(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$$

is undefined, then f cannot be differentiable at $\vec{a}$.

Also, $\vec{F}(\vec{x}) = \langle F_1(\vec{x}), \ldots, F_n(\vec{x}) \rangle$ is differentiable at $\vec{a}$ if and only if all the component functions $F_i(\vec{x})$ are. Therefore, if it is easier, we can check the differentiability of $F_1, \ldots, F_n$ individually.