Math 14
Winter 2009
Practice Problems

1. (a) Complete the following definitions:
    i. The *scalar curl* of a vector field $\vec{F}(x,y) = \langle F_1(x,y),\ F_2(x,y)\rangle$ is:
    ii. A region $E \subseteq \mathbb{R}^3$ is *symmetric elementary* if:
    iii. A curve γ is *smooth* if:

   (b) State Gauss's Theorem.

2. A mug is to be made in the shape of a cylinder, with closed bottom and open top. If the volume held by the mug, when filled to the brim, must be V_0, what should the height of the mug and the radius of its base be in order to have the smallest possible surface area?

3. Use the definition of derivative to prove that if $f(x,y) = xy$ then $f'(2,1) = (1,2)$. You may use any facts about limits that you know.

4. Compute
$$\int_0^1 \int_y^1 \sqrt{x^2+y^2}\,dx\,dy.$$

5. Find the volume of the region
$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1.$$

6. Suppose $f(x,y,z)$ has continuous nonzero derivative everywhere, and the surface $f(x,y,z) = 0$ has a vertical tangent plane at the point (x_0,y_0,z_0). Prove that either the equation $f(x,y,z) = 0$ implicitly defines x as a function of y and z in some neighborhood of (x_0,y_0,z_0), or else the equation $f(x,y,z) = 0$ implicitly defines y as a function of x and z in some neighborhood of (x_0,y_0,z_0),

7. Use Green's Theorem to compute the area of the region
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1.$$

8. Find the surface area of the surface S described in cylindrical coordinates by $r = z$ for $0 \le z \le 1$.

9. Let
$$\vec{F}(x, y, z) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}, z \right).$$
Compute the flux of $\vec{F}$ through the surface S if

   (a) S is the portion of the cylinder $x^2 + y^2 = 1$ between the planes $z = -1$ and $z = 1$, oriented with normal pointing away from the z-axis.

   (b) S is the portion of the sphere $x^2 + y^2 + z^2 = 2$ between the planes $z = -1$ and $z = 1$, oriented with normal pointing away from the z-axis.

10. If $\vec{B}$ is a magnetic field, and a charged particle is moving in this field with velocity $\vec{v}$, there is a resulting force on the particle of
$$\vec{F} = k\vec{B} \times \vec{v},$$
where k is a constant. (Notice that this force is not in the direction of $\vec{B}$.) The magnetic field produced by a constant current on the z-axis is given by
$$\vec{B} = \left(\frac{-y}{x^2 + y^2} \frac{x}{x^2 + y^2}, 0 \right).$$

   (a) Let γ be a piecewise smooth, simple, closed curve in the xy-plane that circles the origin. Show that if a charged particle moves in this field along γ, and $\vec{F}$ denotes the force due to the magnetic field, then
$$\int_\gamma \vec{F} \cdot \vec{T}\, ds = 0.$$

   (b) Suppose γ does not lie in the xy-plane, but circles the z-axis. Do we still have
$$\int_\gamma \vec{F} \cdot \vec{T}\, ds = 0?$$

11. Let
$$\vec{F}(x,y,z) = \left(\frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2},\ z\right).$$
Let γ be the portion of the unit circle in the xy-plane that goes from $(1,0,0)$ to $(0,1,0)$ in a counterclockwise direction, and let $\overline{\gamma}$ be another simple smooth curve that goes from $(1,0,0)$ to $(0,1,0)$, and lies in the portion of the cylinder $x^2+y^2=1$ that lies in the first octant. Show that
$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_{\overline{\gamma}} \vec{F}\cdot d\vec{r}.$$