Math 14
Winter 2009
Preparing for the Second Midterm

At the end of this handout are some practice problems. I have not included examples of problems in which you are asked to give a definition or state a theorem, although there may well be such problems on the exam. Instead, here is a list of important definitions and theorems we have covered so far that you should know and be able to give on an exam.

Things you should be able to define:

Limit (using the ε-δ or the neighborhood definition).

Derivative, and all the concepts needed to define derivative (differentiable, tangent, affine, linear).

Conservative vector field and potential function.

Unit tangent and (in $\mathbb{R}^2$) unit normal vector to a curve.

Gradient, divergence, and curl.

Jacobian determinant.

Path integral, line integral, double integral. (Note that our textbook defines path and line integrals using a parametrization of the curve, and then shows that these integrals are well-defined by showing you get the same answer if you use two different parametrizations, provided, in the case of line integras, they orient the curve the same way. However, the definition of double integral uses Riemann sums.)

Smooth, piecewise smooth, closed, simple, or oriented curve. Intuitive descriptions of closed (the end point is the beginning point), simple (does not cross itself), and oriented (the direction of the curve is specified) are sufficient. (Just for your information, formally, a curve is oriented if there is a unit tangent vector assigned to each point on the curve, and this assignment of unit tangent vectors is continuous.)

Open, closed, bounded, connected, or simply connected (in $\mathbb{R}^2$) region. An intuitive description of simply connected (has no holes) is sufficient. (Recall that a region is connected if any two points in the region can be joined by a path lying in the region, which we used in proving that path independence of line integrals of f implies that f is conservative.)

Any concept I did not mention in the last two paragraphs (such as C^1 or C^2 function) that appears in the statement of any theorem you are responsible for stating.

Theorems you should be able to state carefully and correctly. (You should be able to apply any theorem we have covered.)

The Chain Rule.

The Implicit and Inverse Function Theorems.

The Fundamental Theorem of Line Integrals. (In the book this is called simply "Line Integrals of Gradient Vector Fields.")

The theorem relating various characterizations of conservative vector field. (A version of this is stated in the textbook on page 551. Our version differs slightly. We have that (i), (ii), (iii) are equivalent, and imply (iv), if the domain of $\mathbf{F}$ is any connected open set in $\mathbb{R}^n$. So far, we know only that (iv) implies the others on *simply connected* open sets in the plane $\mathbb{R}^2$.)

Green's Theorem, in both forms (divergence and scalar curl).

Sample Problems

(1.) Express the area of the region of the xy-plane $1 \le x \le \sqrt{4-y^2}$ as an iterated integral. Then evaluate that integral using any of the methods or theorems we have learned. (You can find this area directly by elementary geometric reasoning, without ever looking at an integral. That is not a suitable answer to this problem, although you can check your work that way.)

(2.) Find the average x-coordinate of a point in the region $0 \le x \le 1$, $0 \le y \le \dfrac{1}{x^2+1}$. (Is your answer less than, equal to, or greater than the midpoint of the interval $0 \le x \le 1$? Does this make sense?)

(3.) Sketch the region of integration, and rewrite the integral with the opposite order of integration..

$$\int_{-1}^{1} \int_{|x|}^{x+2} f(x,y)\,dy\,dx$$

$$\int_{0}^{\frac{\pi}{4}} \int_{0}^{\cos y} f(x,y)\,dx\,dy$$

(4.) Evaluate the following line integrals.

$$\int_{\gamma} \langle -y,\, x,\, z\rangle \cdot d\vec{r}$$

where γ is the portion of the helix parametrized by $\vec{r}(t) = \langle \cos t,\, \sin t,\, t \rangle$ for $0 \leq t \leq 2\pi$.

$$\int_\gamma \left\langle y + \sin x,\, x - \ln(\tan^{-1}(y^2+1)),\, \frac{z}{z^2+1} \right\rangle \cdot d\vec{r}$$

where γ is the portion of the helix parametrized by $\vec{r}(t) = \langle \cos t,\, \sin t,\, t \rangle$ for $0 \leq t \leq 2\pi$.

$$\int_\gamma \langle x - y,\, x + y \rangle \cdot \vec{T}\, ds$$

where γ is the unit circle with a counterclockwise orientation.

$$\int_\gamma \langle x,\, y \rangle \cdot \vec{n}\, ds$$

where γ is the curve $y = \sin x$ for $0 \leq x \leq \pi$.

(5.) Determine whether each of the following vector fields is conservative on the domain where it is defined, and if it is, find a potential function.

$$F(x,y) = \left\langle x^2 - y,\, x - y^2 \right\rangle$$

$$F(x,y) = \langle e^x + y\sin(xy),\, \cos y + x\sin(xy) \rangle$$

$$F(x,y) = \left\langle \frac{-y}{x^2+y^2},\, \frac{x}{x^2+y^2} \right\rangle$$

(6.) Evaluate

$$\iint_D x\, dA$$

where D is the region inside the leaf of the four-leaved rose $r = \cos 2\theta$ overlapping the positive x-axis.

(7.) Suppose that

$$\vec{F}(x,y) = \langle P(x,y),\, Q(x,y) \rangle$$

is a C^1 vector field on $\mathbb{R}^2$. Find a condition on the partial derivatives $\dfrac{\partial P}{\partial x}$, $\dfrac{\partial P}{\partial y}$, $\dfrac{\partial Q}{\partial x}$, and $\dfrac{\partial Q}{\partial y}$ sufficient to guarantee that if γ and γ' are any piecewise smooth curves in $\mathbb{R}^2$ with the same beginning and end points, then

$$\int_\gamma \vec{F} \cdot \vec{n}\, ds = \int_{\gamma'} \vec{F} \cdot \vec{n}\, ds.$$

Be sure to justify your answer. (If you need to, you may assume that the two curves do not cross each other, and meet only at their endpoints.)

(8.) Obviously, if a region in the plane is rotated around the origin by some angle its area does not change, but prove it using a change of variable.

(9.) Can every C^2 vector field in $\mathbb{R}^3$ be written as the curl of some other vector field? Prove your answer is correct.

(10.) Use Green's Theorem to evaluate the area of a parallelogram in the first quadrant of the xy-plane whose corners, listed counterclockwise, are $(0,0)$, (a,c), $(a+b,c+d)$, (b,d). (By a claim we have made in class, this area should equal the absolute value of the determinant of a matrix whose columns or rows are the two edges of this parallelogram, or $ad-bc$.)